Empirical Similitude Method for the Functional Test with Rapid Prototypes

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ABSTRACT

Rapid prototyping has the potential to improve the performance of the design process both in cycle time and resources. Such improvements may be realized through the timely visual, ergonomic, and functional information provided by solid freeform fabrication (SFF) parts. Of these information classes, functional information is perhaps the least realized with current technology. A number of technical issues have limited functional testing of SFF parts, including sensor fusion, range of prototyping materials, part size etc. Our focus here concerns the material issues of functional testing, especially the potential differences in prototyping material choices to actual production materials. For example, to derive accurate functional information of non-polymeric products from polymeric rapid prototypes, an improved similitude method that can overcome the distortion of material characteristics is necessary. In this paper, a new similitude method that utilizes specimen test data is introduced. This method develops a mathematical transformation between prototype and product behavior through specimen testing. This transformation replaces the role of the scale factor of the traditional similitude method, and provides a basis for relating prototypes to proposed production parts, even under dependent loading and material conditions. Computational and experimental results of a structural design provide verification of the new method.

1 INTRODUCTION

Design is an iterative process that recursively transforms customer needs into the appropriate set(s) of design parameters within various constraints, and the decision on design parameters are made from virtual and physical models. In order to maintain competitiveness, companies need to provide high-quality products at demanded time within allowable financial resources. Rapid prototypes provide effective visual, ergonomic, and functional information that is similar to the future product in a single physical entity, so it has the potential to accelerate the design process by providing required information with minimal time delay. The derived information is used to verify virtual models or to correct design faults. Among the various information classes, perhaps the functional information is the least realized.

The required effort to build testable models is critical when the geometry of the target system is complex. Fortunately, a new technology, called rapid prototyping (RP) or solid freeform fabrication (SFF), enables effective fabrication of geometrically complex parts. Traditionally, a scaled physical model that can show similar behavior is designed on the basis of the Buckingham Π theorem, and the behavior is correlated to the target system behavior through the derived scaling laws [Kline 75, Baker 91].

It is natural to expect various similitude studies with rapid prototypes, but very little literature exists on that subject [Dornfeld 95, Steinchen 95]. Some of the factors, that are limiting the utilization of rapid prototypes for the functional testing, are: (1) non-similar material characteristics of the rapid prototyping and production materials (e.g., nonlinear material properties); (2) distinct material structure (e.g., non-homogeneous and anisotropic material properties of rapid prototypes); (3) limited material choices; and (4) restrictions on loading conditions (e.g., burning of polymer under combustion). Due to these problems, the testing models fabricated from various rapid prototyping processes may not be suitable to make a precise prediction with traditional similitude methods.

In order to extend the utilization of rapid prototypes for the reliable prediction of the functional behavior of future products, a new similitude method is developed and tested. In comparison to traditional methods, where only the restricted information (the dimension of parameters and variables) is utilized, our new similitude method attempts to derive a local state transformation from the specimen pair(s) whose geometry is simple to fabricate. This novel specimen-based *empirical similitude method* is experimentally and computationally evaluated with three example problems. The new similitude method predicts the functional information well with one or two specimen pairs.

2   Problems in Deriving Functional Information from Rapid Prototypes

There exist two main obstacles that lies in wide utilization of rapid prototypes for functional tests. One originates from the characteristics of rapid prototypes, and another comes from the restrictions on the traditional similitude methods.

2.1   Current Status of Rapid Prototyping Techniques

The last decade has seen the emergence and continuous advancement of various rapid prototyping techniques, and at least 20 companies commercialized diverse rapid prototyping systems in Europe, Japan, and USA. Most of the systems can fabricate geometrically complex prototypes within 50 hours, where polymers are the most popular base material [Aubin 94]. As rapid prototyping emphasizes the dramatically reduced fabrication cost and time, the loss or distortion of some information classes is inevitable. The limited capacity (fabricable prototype size) and material choices of the rapid prototyping system are main factors that prevents full scale tests in various situations.

2.2   Limitations of Traditional Similitude Methods

Traditional similitude methods, which are based on the Buckingham Π theorem, have been widely used in order to transform the behavior of physical prototypes to that of target systems [Kline 75, Baker 91]. The methods simply correlate (by multiplying scale factors) the corresponding behavior through a dimensionless parameter group. However, these methods have following limitations to make precise prediction of product behavior through rapid prototypes.

(i)   When two systems are governed by equations with different configurations, the prediction is erroneous. The prediction of the deformation of a structure with small deflection from a structure with large deformation is the typical example of this type.

(ii) The system parameters, that are related to the system behavior of interest, should be constant. However, the dependency of material parameters (e.g., thermal conductivity) on the system behavior (e.g., temperature) may not be negligible. In addition, the spatial distribution of material properties of the prototype and product are expected to be distinct, due to the unique fabrication schemes.

(iii) All corresponding dimensionless parameters of two systems should be identical in the traditional methods. Due to the limited material choices and difficulties in generating exact boundary conditions, the corresponding dimensionless parameters cannot always made to be identical.

3   New Specimen-Based Empirical Similitude Method

The above mentioned limitations of traditional similitude methods seem to come from the utilization of restricted information, i.e., the dimensions of parameters and variables. In comparison, our new similitude method attempts to derive extended information from the geometrically simple specimen pair(s) (one from the rapid prototyping and another from the production process) that can be easily fabricated.

The new specimen-based empirical similitude method, that may overcome the limitations of traditional similitude methods, assumes that there exists a consistent local transformation between the corresponding *n*-dimensional sub-state vectors[1] of two systems *only if* the geometry is similar. Based on this concept, the new similitude method derives a local state transformation matrix from the measured states of specimen pairs, and the derived state transformation matrix replaces the role of the scale factor in traditional methods. Then, the unknown states of the future-product are estimated from this derived state transformation matrix and the measured states of the rapid prototype. In Figure 1, the overall concept of the new similitude method is described, the state vectors are defined, and two plausible methods that realizes the concept are introduced in following subsections.

The state transition due to the material change is abstracted from the specimen pair. As the state variation due to the geometrical variation is represented with the measured state vector of the target model, one can estimate the state vector of the target product.
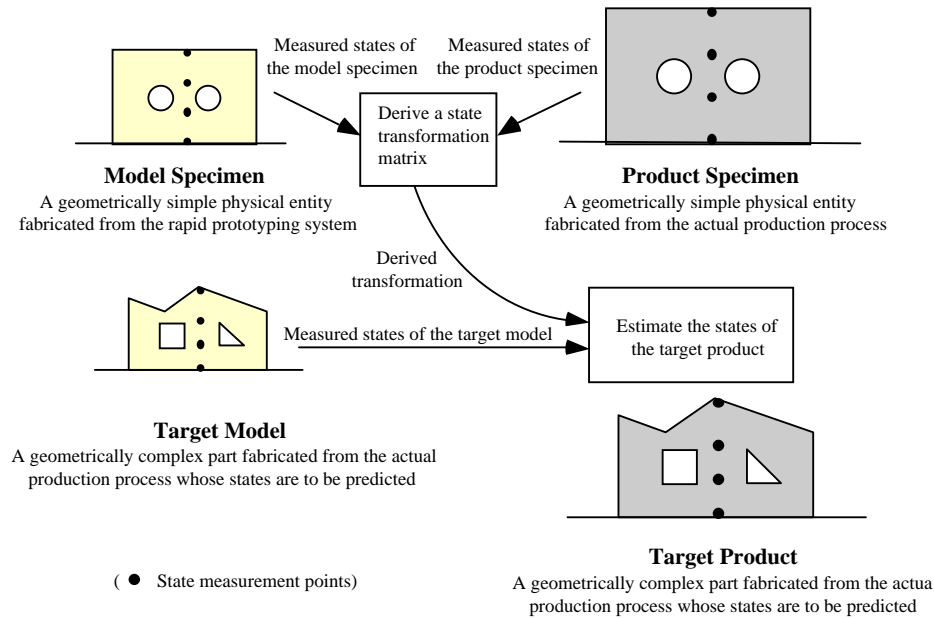
Measured states of the model specimen   Measured states of the product specimen

Derive a state transformation matrix

Model Specimen
A geometrically simple physical entity fabricated from the rapid prototyping system

Product Specimen
A geometrically simple physical entity fabricated from the actual production process

Derived transformation

Measured states of the target model

Estimate the states of the target product

Target Model
A geometrically complex part fabricated from the actual production process whose states are to be predicted

Target Product
A geometrically complex part fabricated from the actua production process whose states are to be predicted

(● State measurement points)

Figure 1: Overall Concept of the New Similitude Method.

Definition of State Vectors
- *Sub-state vector of the target product,* $\mathbf{x}^{tp} = (x^{tp}_1 \ x^{tp}_2 \ \cdots \ x^{tp}_k)^T$: *k by 1* vector that is composed of unknown physical values of interest (e.g., temperature) on at *k* local points of target product
- *Sub-state vector of the target model,* $\mathbf{x}^{tm} = (x^{tm}_1 \ x^{tm}_2 \ \cdots \ x^{tm}_k)^T$: *k by 1* vector that is composed of measured physical values of interest at *k* corresponding local points of the target model
- *Sub-state vector of the i-th product specimen,* $\mathbf{x}^{psi} = (x^{psi}_1 \ x^{psi}_2 \ \cdots \ x^{psi}_k)^T$): *k by 1* vector that is composed of measured physical values of interest at *k* corresponding local points of the product specimen
- *Sub-state vector of the i-th model specimen,* $\mathbf{x}^{msi} = (x^{msi}_1 \ x^{msi}_2 \ \cdots \ x^{msi}_k)^T$: *k by 1* vector that is composed of measured physical values of interest at *k* corresponding local points of the model specimen

3.1 Similitude with Absolute Linear Transformation

The unknown relationship between $\mathbf{x}^{tp}$ and $\mathbf{x}^{tm}$ is approximated from the relation(s) of $\mathbf{x}^{psi}$ and $\mathbf{x}^{msi}$. As an initial attempt, the relationship is assumed as a discrete linear transformation.

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[1] The sub-state vector is defined as n by 1 vector whose components are composed of measurable quantities to be predicted, on a sub-domain D_s in local coordinate space, where $D_s \subset D$ and D is entire domain of the interest.

Let's consider a linear transformation matrix **T** that satisfies following equation

$$\mathbf{x}^{ps1} = \mathbf{T} \cdot \mathbf{x}^{ms1}$$ (1)

where **T** is a *k by k* matrix.

In traditional similitude methods, the **T** is a diagonal matrix whose diagonal terms have identical state scale factor s. The scalar scale *s* is a real constant, and it is a function of *N* design parameters, DP_i (e.g., characteristic length, material constants), and *M* loading parameters, F_j, as shown below.

$$s = \frac{x_i^p}{x_i^m}$$

$$= \left[\left(\frac{DP_1^p}{DP_1^m} \right)^{R_1} \cdot \left(\frac{DP_2^p}{DP_2^m} \right)^{R_2} \cdots \left(\frac{DP_N^p}{DP_N^m} \right)^{R_N} \cdot \left(\frac{F_1^p}{F_1^m} \right)^{R_{N+1}} \cdot \left(\frac{F_2^p}{F_2^m} \right)^{R_{N+2}} \cdots \left(\frac{F_M^p}{F_M^m} \right)^{R_{N+M}} \right]^{R_{N+M+1}}$$ (2)

where R_i is a real number, and superscript *p* and *m* denote the product and model respectively. Once all the DP_i's and F_j's are decided, *s* becomes a constant real value, and the ratio of corresponding states of the product to model is same to the state scale factor *s*, at any point.

In comparison, the new similitude method expects improved similitude results by empirically deriving the matrix **T** that satisfies equation (1). The empirically derived transformation matrix T is generally not a diagonal matrix, and the elements, t_{ij}'s, of the matrix **T** can be interpreted as the weighting factor on the *j-th* state of $\mathbf{x}^{ms1}$ to predict the *i-th* state of $\mathbf{x}^{ps1}$. So, the state of the product specimen is not a scaling of only corresponding state of the model specimen but a balance of the considered neighboring states.

If states of *n* specimen pairs are measured, then the transformation matrix **T** should satisfy the following equation,

$$\left[\mathbf{x}^{ps1} \; \mathbf{x}^{ps2} \; \cdots \mathbf{x}^{psn} \right] = \mathbf{T} \cdot \left[\mathbf{x}^{ms1} \; \mathbf{x}^{ms2} \; \cdots \mathbf{x}^{msn} \right].$$ (2)

It is desired to minimize the number of specimens *n*, so the number of columns is usually lesser than that of rows, i.e., the size of the considered states. So, the number of unknowns are lesser than that of given equations. When the $\left[\mathbf{x}^{ms1} \; \mathbf{x}^{ms2} \; \cdots \mathbf{x}^{msn} \right]^{-1}$ does not exist, one way to find the **T** that satisfies equation (2) is to use the pseudo-inverse matrix [Strang 1988] as follows.

$$\mathbf{T} = \left[\mathbf{x}^{ps1} \; \mathbf{x}^{ps2} \; \cdots \mathbf{x}^{psn} \right] \cdot \left[\mathbf{x}^{ms1} \; \mathbf{x}^{ms2} \; \cdots \mathbf{x}^{msn} \right]^{+}$$ (3)

where A^+ is the pseudo-inverse matrix of A.

Once the transformation matrix **T** is obtained, the state vector of the target product $\mathbf{x}^{tp}$ is derived from the following equation,

$$\mathbf{x}^{tp} \cong \mathbf{T} \cdot \mathbf{x}^{tm}.$$ (4)

As the transformation matrix **T** characterizes the state transition due to the material variation, the matrix for both specimen and target pair may be identical if the corresponding points' location in specimens and targets are well decided. If not specially stated, this approach is used as the new similitude method in Section 4.

3.2 Relative Linear Transformation

The previous method assumes that the derived state transformation matrix **T** is independent of the geometry. However, the transformation may be dependent on the geometry. So, a method, that involves

an alternative linear transformation $\delta\mathbf{T}$ that relates the deviation of states with respect to the reference specimen, is introduced as an adherent of the first method.

In this method, the state vectors of the first specimen pair, $\mathbf{x}^{ps1}$ and $\mathbf{x}^{ms1}$, are considered as reference vectors. The variation of the state vectors of the second specimens are defined as

$$\Delta\mathbf{x}^{ps} = \mathbf{x}^{ps2} - \mathbf{x}^{ps1}$$
$$\Delta\mathbf{x}^{ms} = \mathbf{x}^{ms2} - \mathbf{x}^{ms1}$$

$$(5)$$

where $\mathbf{x}^{ps2}$ and $\mathbf{x}^{ms2}$ are state vectors of the second specimen pair. Similar to the previous method, the linear transformation $\delta\mathbf{T}$ that maps the state variation instead of the state itself, is derived from the following equation.

$$\delta\mathbf{T} = \Delta\mathbf{x}^{ps} \cdot (\Delta\mathbf{x}^{ms})^{+} \tag{6}$$

Assuming that $\delta\mathbf{T}$ is constant (instead of $\mathbf{T}$), i.e., the following equation is satisfied

$$(\mathbf{x}^{tp} - \mathbf{x}^{ps1}) = \delta\mathbf{T} \cdot (\mathbf{x}^{tm} - \mathbf{x}^{ms1}), \tag{7}$$

$\mathbf{x}^{tp}$ can be determined from

$$\mathbf{x}^{tp} = \mathbf{x}^{ps1} + \delta\mathbf{T} \cdot (\mathbf{x}^{tm} - \mathbf{x}^{ms1}) \tag{8}$$

as $\mathbf{x}^{ps1}$, $\mathbf{x}^{tm}$ and $\mathbf{x}^{ms1}$ are to be measured.

4 EVALUATION of the NEW SIMILITUDE METHOD

Several experimental and numerical tests are carried out to test the validity of the new specimen based empirical similitude method.

4.1 Beam Experimentation

(a) Specimen Pair   (b) Target Beams
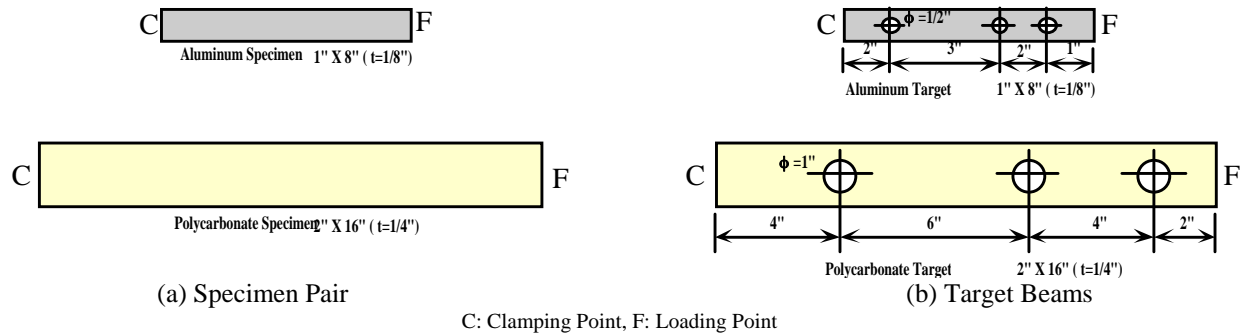C: Clamping Point, F: Loading Point
Figure 2: Aluminum and Polycarbonate Specimens and Targets

In the beam experimentation, the deflection of the aluminum target beam at 1" (25.4 mm) and 7" (177.8 mm) from the clamping point is predicted. In **Figure 2,** the geometry of the specimens and target beams are shown (excluding the clamping and loading zone). The force on the polycarbonate specimen and target is scaled down to half of that applied to aluminum specimen and target, in order to emphasize the geometrical non-linearity due to the large deflection. For the convenience of the measurement, the LVDT is fixed to the same location without following the point of the beam when the large deflection is measured.

Two possible sources of dissimilarity, that obstruct the reliable prediction of the deflection of the aluminum target with traditional similitude methods, are: (1) the nonlinear stress-strain characteristics of the polycarbonate; and (2) the geometrical non-linearity due to the large deflection.
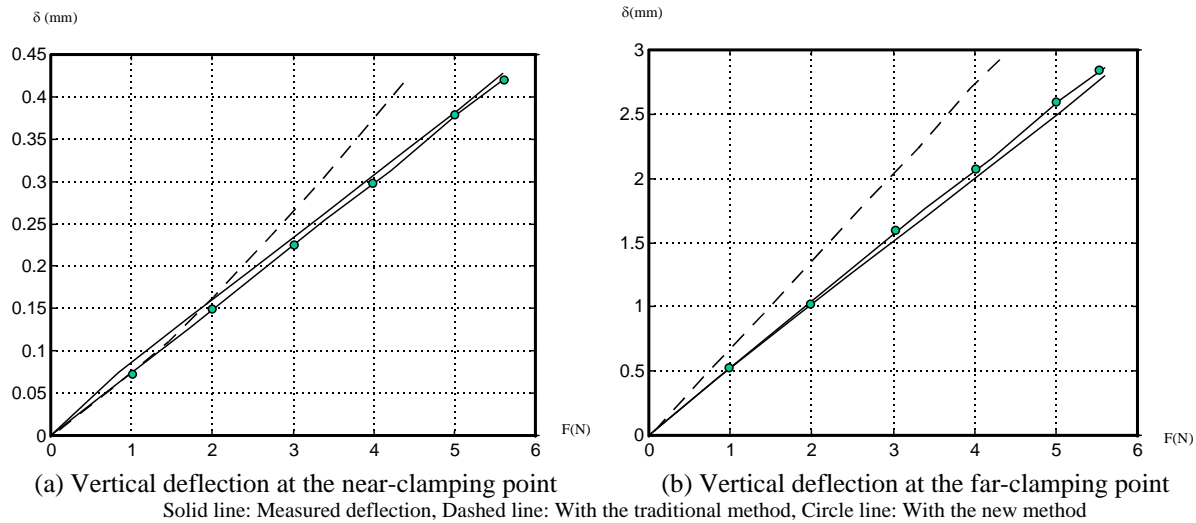

(a) Vertical deflection at the near-clamping point   (b) Vertical deflection at the far-clamping point

Solid line: Measured deflection, Dashed line: With the traditional method, Circle line: With the new method

Figure 3: The Vertical Deflection of the Aluminum Target

The vertical deflection of the aluminum target beam is predicted at two points (near-clamping point: 1" (25.4 mm) from the clamping point, far-clamping point: 7" (177.8 mm) from the clamping point) with the traditional and new similitude method. As shown in **Figure 3**, the similitude result with the new method shows remarkable matching with the actually measured deflection of the aluminum target beam in comparison to the traditional similitude method.

4.2 Computational Evaluation

Two finite element (FE) simulations were performed with ANSYS to test the validity of the new similarity method in various situations. The assumed material properties for the FE simulations are as follows.

Material	Young's modulus E (GN/m^2)	Density (kg/m^3)	Thermal conductivity k (W/m·K)	Thermal expansion coefficient α (/K)
Aluminum	69.0	2720.0	170.0	0.23E-4
Polycarbonate	4.0 ~ 7.0	1000.0	0.193	0.7E-4

Table 1: Material Properties for the FE Models

4.2.1   Static Structural Problem

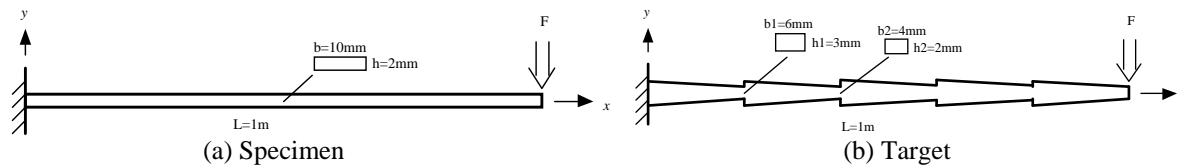(a) Specimen   (b) Target

Figure 4: Specimen and Target Beam

The problem is to predict the deflection of the aluminum target on the basis of the specimen testing results and the deflection of the polymer target. The geometry and force scale factor is set to one, and the specimen and target geometry is shown in **Figure 4**. The Young's modulus of aluminum is assumed as a constant, and the polycarbonate is modeled as stress softening material whose Young modulus varies from 7 to 4 GN/m^2. In addition, the aluminum beams are assumed to show small deflection (geometrically linear), and deflection of the polycarbonate beam is large (geometrically nonlinear)
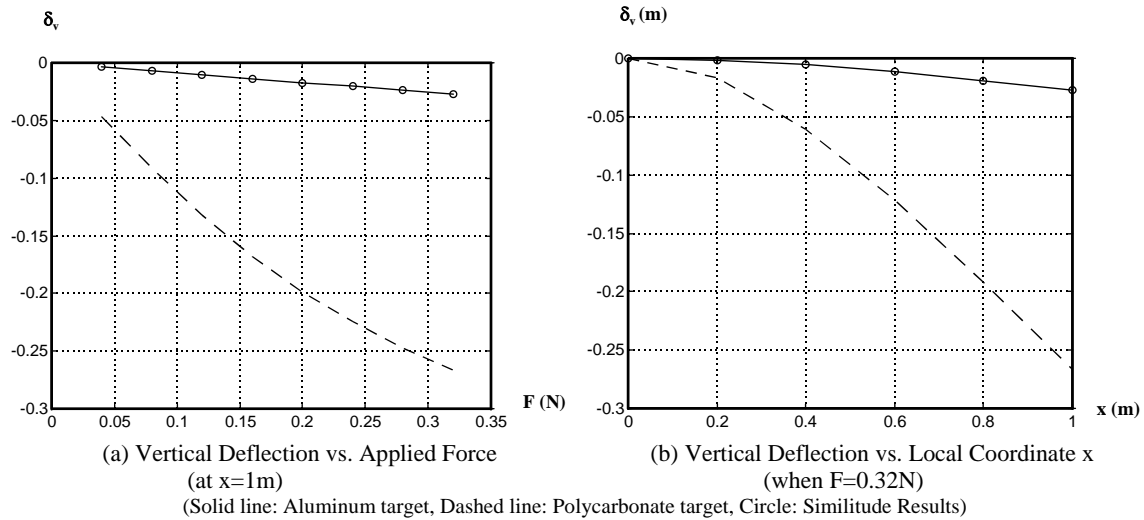
(a) Vertical Deflection vs. Applied Force
(at x=1m)

(b) Vertical Deflection vs. Local Coordinate x
(when F=0.32N)

(Solid line: Aluminum target, Dashed line: Polycarbonate target, Circle: Similitude Results)

Figure 5: Similitude Results of the Beam Deflection

As shown in **Figure 5**, the new method predicts the deflection of the aluminum target beam within 1% error. The results on the angular displacement shows the similar results, even though it is not shown here due to the space limitation. In this simulation, the new method demonstrated its ability to map the behavior of two systems whose governing equations take different forms. Besides, the new similitude method successfully predicts the behavior of the target beam when the Young's modulus is not constant.

4.2.2   Steady State Temperature and Thermal Stress

The thermo-structural problem is introduced as an example to demonstrate the feasibility of the new similitude method when some of the dimensionless parameters are not kept to be identical. The geometry and boundary conditions of the considered specimen pairs and the target system are shown in **Figure 6**. The boundary condition at the outer surface of the polymer and aluminum specimens and targets are set to be same, assuming that one cannot easily control them (e.g., natural convection). The difficulty in using polymeric rapid prototypes for the thermal similitude lies in the small thermal conductivity constant, and the low sintering temperature. Considering these material characteristics, the boundary conditions at the inner surface is determined as shown in **Figure 6**.
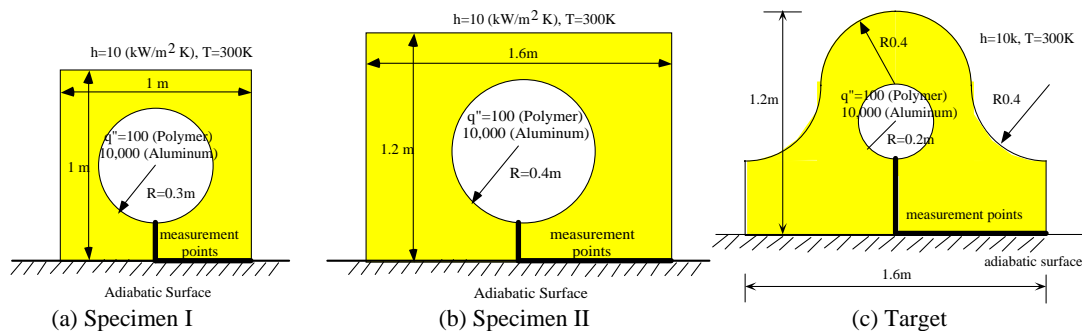

(a) Specimen I   (b) Specimen II   (c) Target

Figure 6: Thermal Specimens and Targets

In the two dimensional problems, the difficulty lies in deciding corresponding measurement points. In this example problem, characteristic points are defined as the point on the boundary surface with same boundary conditions, and the characteristic lines are defined as the line that connects two characteristic points. Once the characteristic lines are determined, the lines are linearly divided to find out the corresponding measurement points. The steady state temperature are measured along the two characteristic lines as shown in **Figure 6**.

In **Figure 7**, the accuracy of the predicted temperature increases as the number of considered specimens increases. Even though the improved prediction accuracy is expected by considering more specimens, alternative method that utilizes the relative state transformation (see Subsection 3.2) has been applied and tested. In **Figure 8**, the alternative method shows noticeable improvement of the prediction accuracy with the same number of specimens.
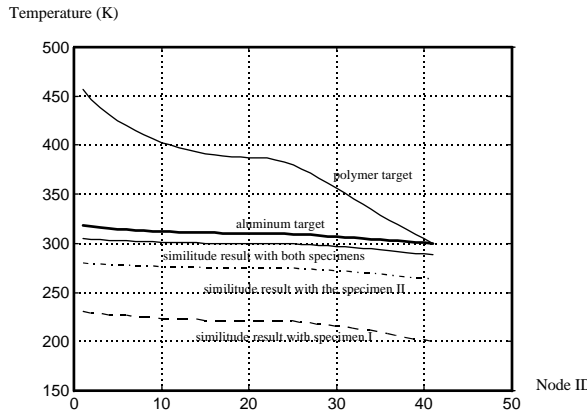
Figure 7: Similitude Results with the Absolute Transformation
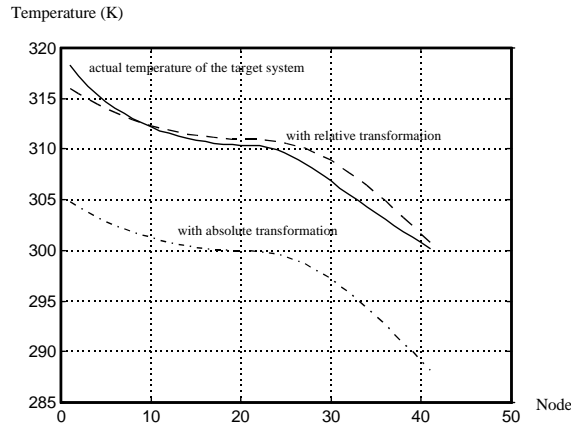
Figure 8: Similitude Result with the Absolute and Relative Transformation

In comparison to the similitude error in the temperature, the accuracy of the predicted thermal stresses (normal and shear) is remarkable as shown in **Figure 9**.
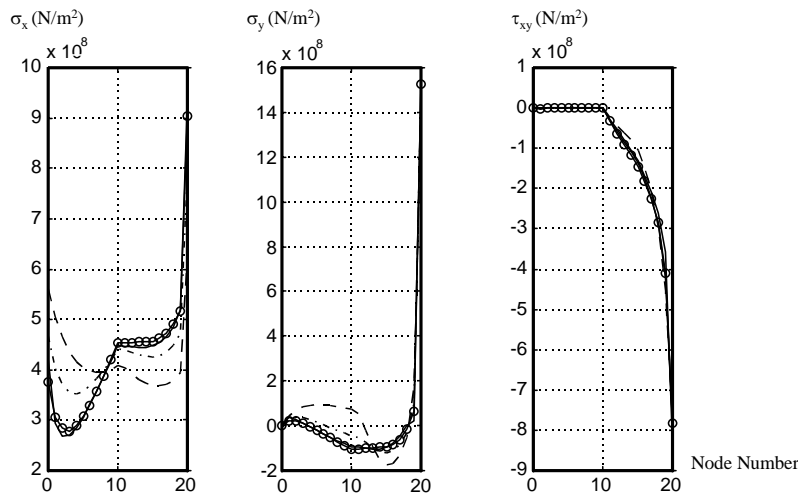
Solid line: Aluminum Target, Circle: Similitude results with 2 specimens
Dashed line: Similitude Results with Specimen 1, Dashdot line: Similitude Results with Specimen 2
Figure 9: Thermal Stress Similitude Results

5   Conclusions

The new specimen-based empirical similitude method predicts the various functional behavior (deformation, pure stress, temperature, and thermal stresses) of all considered example problems that cannot be accurately predicted with traditional similitude methods. The new similitude methods with absolute and relative transformation have pros and cons their own, and a better transformation may be

derived under the same concept of this paper − utilization of specimens to derive a consistent state transformation.

Other problem domains (e.g., structural dynamics) will be tested to define the feasible application areas. For the more realistic evaluation of the new similitude method, experimentation with SLS prototypes will be carried out in the near future.

6   Acknowledgments

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(Key Words)

similitude method, rapid prototype, industrial design process, functional test, distorted model,

state transformation, scale factor, prototyping material