Representing Imprecision in Engineering Design – Comparing Fuzzy and Probability Calculus *

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Abstract

A technique to perform design calculations on imprecise representations of parameters using the calculus of fuzzy sets has been previously developed [25]. An analogous approach to representing and manipulating *uncertainty in choosing among alternatives* (design imprecision) using probability calculus is presented and compared with the fuzzy calculus technique. Examples using both approaches are presented, where the examples represent a progression from simple operations to more complex design equations. Results of the fuzzy sets and probability methods for the examples are shown graphically. We find that the fuzzy calculus is well suited to representing and manipulating the imprecision aspect of uncertainty in design, and probability is best used to represent stochastic uncertainty.

1 Introduction

In order to contribute to a *theory of engineering design* [1], it is useful to formulate a hypothesis which will lead to a better understanding of design practice by developing morphologies of design, or by improving current analytical or computational design procedures. Our research primarily falls in the latter of these categories, where the main objective is to provide methods and tools for the decision-making aspect of *preliminary engineering design*.

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A previous publication [25] describes the first steps toward the realization of this objective. That paper presents a method for modeling the *uncertainty in choosing among alternatives* found in preliminary design (which we define as *design imprecision*), using the fuzzy calculus. It also develops analytical and computational procedures for carrying out calculations with this type of uncertain (imprecise) design parameter. Much of the terminology used to describe the method, as well as applications and examples, can be found in [25], along with [24]. This paper will develop a similar technique using multi-valued probability logic to represent and manipulate imprecise parameters, and will compare it with the authors' fuzzy calculus method.

Other authors have compared the fuzzy and probability calculus as they apply to decision support [5, 7], logic [14, 28, 29], and analysis of risk [4, 6]. This paper explores the application of these two calculi to representing and manipulating *imprecision* in (preliminary) engineering design.

1.1 Imprecision in Engineering Design

Engineering design is a process comprised of many recognizable stages: evaluation of functional requirements and constraints; concept generation; simplification of design alternatives; analysis and judgement of feasibility; concept evaluation and refinement; embodiment design; detail design; etc. (Dixon [10, 11], Pahl and Beitz [20]). Once several concepts have been generated, we view this process as one of reducing the uncertainty with which each design alternative is described. At the preliminary stage, the designer is not certain what values will be used for each design parameter. For example, during the design of a gearing system for the transfer of power from one shaft to another, the various dimensions of the gear teeth pitch, radii, lengths, etc. are not known with any certainty for early configurations. Instead, these dimensions are usually given in terms of "approximate" (imprecise) or "probable" values within a certain range. Uncertainty of this type is comprised of two distinct forms of uncertainty: *design imprecision* (uncertainty in choosing among alternatives) and *stochastic uncertainty* (usually pertaining to manufacturing and measurement limitations).

An imprecise parameter in preliminary design is a parameter which may potentially assume any value within a possible range because the designer does not know, *a-priori*, the final value that will emerge from the design process. The designer is free (at least at the beginning of the design process) to choose the value for this parameter. The nominal value of a length dimension is an example of an imprecise parameter. Even though the designer is (intellectually) uncertain about the final value of a design parameter, he or she usually has a preference or desire for choosing certain values over others, even before the design is analyzed and compared to the functional requirements. We use this preference to quantify the imprecision with which design parameters are known (in the preliminary phase of engineering design). Imprecision is represented and manipulated in this way as a basis for decision-making during the progression of the design process from imprecise descriptions of design alternatives to a precise physical device or process.

Stochastic uncertainty, on the other hand, arises from a lack of exact knowledge of a parameter due to some process the designer has no direct control or choice over. A manufacturing tolerance is an example of such an uncertainty. Stochastic uncertainties are usually represented and manipulated using probability [9, 16].

The level of imprecision in the description of design elements is typically high in the preliminary phase of engineering design. As the design process proceeds, the imprecision with which each design parameter is known is reduced (although the stochastic uncertainty usually remains). An illustration of this is shown in Figure 1. This figure suggests general trends of the magnitude of uncertainty commonly observed during the engineering design process. Wide variations from the monotonically decreasing curves will frequently exist, because of the iterative nature of the design process.

This paper concentrates on comparing two different calculi (fuzzy and probability) for representing and manipulating *design imprecision*. The three uncertainties shown in Figure 1 should be kept distinct, as the interpretation of these uncertainty data is different. This paper addresses the question: "Which calculus is the best for representing and manipulating *design imprecision* data?", and compares the fuzzy calculus with the most commonly used two-dimensional parameter calculus: probability.

1.2 Relevance to Engineering Design

Conventional computer-aided design methods require highly precise descriptions in order to operate, making them difficult or impossible to use during the early design process. The primary characteristic of the preliminary stage of engineering design is imprecise descriptions of the design alternatives. When a designer is making sketches (on the "back of the envelope") he or she may, for example, represent a complex beam cross-section by a rectangle, for the purposes of comparison of one design alternative to another. This is a simplified, or imprecise representation of the beam. This approach to design is frequently used because the design is insufficiently refined to permit more precise representation at this stage. As the design process (iteratively) proceeds, the precision with which each aspect of each design alternative is represented is increased.

This research has focused on developing a method for representing and manipulating imprecise descriptions of designs to provide the designer with more information to compare alternatives during the preliminary design phase. Naturally the designer may also wish to determine the effect of stochastic uncertainty on a design alternative. This should be done with the probability calculus, and a method for (separately) manipulating these two design uncertainties in parallel is presented elsewhere [26, 27].

1.3 Terminology

In a previous publication [25], terminology relating to the design process was introduced. A summary of those definitions follows.

Design Parameters (DPs) are variables to be determined during the design process.

Performance Parameters (PPs) have specified target values.

Functional Requirements (FRs) are the specified target values for each of the PPs. Functional requirements are not restricted to the equality and inequality forms normally encountered. Qualitative statements, such as "maximize heat dissipation", or functional relationships relating certain imprecise design parameters to the functional requirement, *e.g.*, "brake torque must be less than or equal to three-fourths of the force on one wheel multiplied by the wheel radius", may be specified.

Performance Parameter Expressions (PPEs) contain (usually in equation form) the relationship between the DPs and the PPs.

1.4 Objectives

Other methods (*e.g.*, probability [15, 22], interval analysis [19], sensitivity analysis [2, 3, 18], the Taguchi method [8, 23], etc.) have previously been applied to representing and manipulating the subjective or uncertain aspects of engineering problems. Probability calculus, in particular, is well known to be appropriate for manipulating one type of uncertainty (what we call *stochastic uncertainty*). As a comparison with these methods, we present an approach here using the *probability* calculus to manipulate the *design imprecision* (not stochastic uncertainty) in a way analogous to our fuzzy calculus method described in [25].[1] The next section of this paper will summarize the fuzzy calculus approach in order to provide a basis for comparison. The subsequent section will develop a probabilistic alternative to the theory of imprecise calculations. Examples using both methods are then presented, followed by a discussion of the major differences. It will be shown that the probability method is applicable for calculations with *stochastically uncertain* parameters, and the fuzzy calculus is well suited for representing and manipulating imprecise design descriptions. It will also be shown that the probability method is not as well-suited as the fuzzy calculus for representing and manipulating imprecise design descriptions.

2 Summary of the Fuzzy Sets Approach

This section will outline the fuzzy calculus method for efficiently representing and manipulating imprecise input design parameters. Operation rules for fuzzy arithmetic will be briefly discussed, followed by the interpretation of imprecise parameters in engineering design based upon the use of such operations. A description of the analytical and numerical application of the fuzzy calculus approach concludes this section.

2.1 Prior Work

Before we present the fuzzy operations, a brief overview of our approach will be described. The *semi-automated* design method [25], developed to aid the designer in making deci-

[1] Naturally, we continue to use probability to represent and manipulate *stochastic* uncertainty throughout our method.

sions in the the preliminary phase of design, has the following objectives: (1) to determine the performance parameters for alternative designs including the designer's subjectivity concerning the choice of input design parameter values; (2) to determine the interaction of the input parameters with respect to the output (coupling and importance of inputs); (3) to rate the performance of each alternative design; and (4) to compare the major differences between alternative designs. Computational efficiency of the design technique is a final objective.

The method is briefly summarized as follows. The designer determines the approximate or exact relationships governing the design alternatives under consideration, thus specifying the performance expressions of the design. Imprecise inputs to the expressions, in the context of choosing among alternative values, are selected by representing and interpreting them as fuzzy numbers. These fuzzy numbers represent the set of design parameters, and are a subjective expression of the designer's preference (desire) for choosing different values of the input parameters.

Once the input parameters and performance parameter expressions have been determined, an analytical or numerical method is used to solve for the imprecise performance parameter results.[2] These results are compared to the functional requirements of the design, and the *backward path* [25] can be utilized to determine combinations of input parameters which will produce a particular desired output (if the inputs that the designer desired most produce results that do not meet the requirements). Though at the preliminary stage the designer is usually most interested in determining how well a design alternative will perform, compared to the other alternatives, and in comparison with the functional requirements, he or she is also quite likely to want to know which input design parameter values produce which output performance parameter values. The designer has specified preferences (as above) for design parameter values and ultimately will have to specify particular design parameter values, once the design is complete. Therefore providing feedback to the designer on which input design parameter values satisfy the functional requirements (with the highest preference values) is vital to any design method.

Relative importance, interaction, and coupling of the inputs can then be determined through the application of the γ-level measure [25].

2.2 Fuzzy Arithmetic Operations

In his seminal paper [30], Zadeh presents the concept of a *fuzzy set* as "a class of objects with a continuum of grades of membership". From the theoretical developments and applications which have appeared subsequent to this original work, a sub-area of research as been devoted to the concept of a *fuzzy number* which concerns exclusively the set of real numbers $\Re$. Kaufmann and Gupta [17] define a fuzzy number as "a fuzzy subset of

[2]Though most of our research and examples have used parametric descriptions of designs, with analytical expressions for the performance parameters, nothing in principle limits the technique to these areas. Qualitative preferences (*e.g.*, aluminum is preferable to steel) can be included, as well as numerical methods to calculate performance parameters.

$\Re$ that is convex and normal". Using this concept, and defining imprecise input design parameters as fuzzy numbers, the *extension principle* of Zadeh [31] can be directly applied to design computations, thus extending algebraic operations on real numbers to the fuzzy domain.

Specifically, let the imprecise input parameters $\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_N$ be defined in the universes $X_1, X_2, \ldots, X_N$, respectively. The mapping from $X_1 \times \cdots \times X_N$ to a universe Y is defined by a function f such that $y = f(x_1, \ldots, x_N)$. The extension principle then implies that for an imprecise performance parameter $\tilde{D}$ on Y which is induced from $\tilde{U}_1, \ldots, \tilde{U}_N$ through f, the resulting membership function is:

$$\mu_{\tilde{D}}^{(y)} = \sup_{\substack{x_1,\ldots,x_N \\ y=f(x_1,\ldots,x_N)}} \min(\mu_{\tilde{U}_1}^{(x_1)}, \ldots, \mu_{\tilde{U}_N}^{(x_N)}). \tag{1}$$

The ordinary binary operations and other function operations (extended trigonometric functions, etc.) can be derived from this principle in this max-min form. For example, the addition of two imprecise parameters $\tilde{U}_1$ and $\tilde{U}_2$ can be written as

$$\mu_{\tilde{U}_1 \oplus \tilde{U}_2}(y) = \bigvee_{y=u_1+u_2} (\mu_{\tilde{U}_1} \wedge \mu_{\tilde{U}_2}),$$

where $\oplus$ denotes extended addition and $\vee$ and $\wedge$ denote max and min, respectively. Equivalently, extended addition can be expressed in the intervals of confidence, level of presumption, and α-cut form of Kaufmann and Gupta [17]:

$$\tilde{U}_{1,\alpha} \oplus \tilde{U}_{2,\alpha} = [u_{1,1}^{(\alpha)} + u_{2,1}^{(\alpha)}, u_{1,2}^{(\alpha)} + u_{2,2}^{(\alpha)}], \tag{2}$$

where $\tilde{U}_{i,\alpha} \in [u_{i,1}^{(\alpha)}, u_{i,2}^{(\alpha)}]$. The method for applying the extended addition and the other extended operations in the design domain can be found in [25]. Note the change in nomenclature from Equation 1 (μ) to Equation 2 (α), reflecting the change from the general extension principle to the α-cut method.

2.3 The Fuzzy Calculus Interpretation

Fuzzy numbers, defined as having an interval of confidence and levels of presumption, provide an effective means by which input design parameters can be represented and interpreted. In general, a range of real numbers might be used to represent a parameter approximately, in the style of interval analysis. We represent imprecision by a range and a function defined on this range (normalized between zero and one) to describe the desirability or preference of using particular values. This function is a quantification of design preference, and not the customary notion of membership in a symbolically labeled fuzzy set, which usually denotes vagueness in meaning. The more the designer desires to use an input value, the higher its function values (α). In this way, parameters whose values are not known precisely can be presented, and the designer's experience and judgement can be incorporated into the design evaluation [25].

After calculating the result of a performance expression using the design parameters as described above, and the authors' method described below, the output performance parameter will also be in the form of a function, with an output interval encompassing all possible output performance values and with a preference function ranging between zero (0) and one (1). If the value of the output interval where the preference value equals one (1) satisfies the functional requirement, the design can proceed without further analysis of this performance parameter, allowing the designer to concentrate on other functional requirements. However, if the functional requirement is not met, the output function can be examined to determine the preference for an acceptable range of output. From this information, a range for each input can be found that corresponds to the acceptable output performance range. These ranges are found without further calculation by use of the *backward path*.

2.4 Analytical and Numerical Applications

For a given design performance expression $z = f(u_i)$, where z is the performance parameter and the u_i are the design parameters, the fuzzy output $\tilde{z}$ can be determined analytically by applying the α-cut form of the extended operations as presented earlier (Kaufmann and Gupta [17], Equation 2). Such an application requires that the input parameters be represented algebraically, where the result after algebraic manipulation is $\tilde{z}_\alpha$ which is a function of the preference values α. Substituting a value for α will give an output corresponding to that level of desirability. (The analytical application for an example design equation can be found in [25], and the general approach for the analytical application of fuzzy arithmetic can be found in [13, 17].)

Even for a modest number of design parameters, the analytical fuzzy calculus method for calculating imprecise performance parameters is impractical for computer-assisted design applications, due to algebraic complexity. A discrete numerical method, such as the Fuzzy Weighted Average (FWA) algorithm [12] and its extensions, was presented in [25] in order to meet this need for computational efficiency. FWA approximates the analytical approach by discretizing the functions of the input fuzzy numbers into a prescribed number of α-cuts. Performing interval analysis for each α-cut and combining the resultant intervals results in the discretized output. A combinatorial interval analysis technique is also included in order to avoid the problem of the multiple occurrence of variables for division and multiplication in the algebraic equation expression. A condensed version of the algorithm from [12] has been provided below (where the terminology has been changed to reflect the application to design calculations).

For N real imprecise design parameters, $\tilde{U}_1, \ldots, \tilde{U}_N$, let u_i ($i \in [1, N]$) be an element of $\tilde{U}_i$. Given a performance parameter represented by the expression $z = f(u_1, \ldots, u_N)$ $\forall u_i \in \tilde{U}_i$ respectively, let $\tilde{Z}$ be the fuzzy output of the expression. The following steps lead to the solution of $\tilde{Z}$.

1. For each $\tilde{U}_i$, discretize the preference function into a number of α values, $\alpha_1, \ldots, \alpha_M$, where M is the number of steps in the discretization.

2. Determine the intervals for each parameter $\tilde{U}_i$, $i = 1, \ldots, N$ for each α-cut, α_j, $j = 1, \ldots, M$.

3. Using one end point from each of the N intervals for each α_j, combine the end points into an N-*ary* array such that 2^N distinct permutations exist for the array.

4. For each of the 2^N permutations, determine $z_k = f(u_1, \ldots, u_N)$, $k = 1, \ldots, 2^N$. The resultant interval for the α-cut, α_j, is given by

$$Z_{\alpha_j} = [min(z_k), max(z_k)].$$

For N fuzzy inputs and M discrete points on the preference function, the algorithmic complexity of the FWA implementation is of order

$$H \sim M \cdot 2^{N-1} \cdot \kappa \tag{3}$$

where H equals the number of operations and κ equals the number of multiplications and divisions in $f(u_i)$.

In general the FWA, when combined with an appropriate user interface, provides a means for carrying out real-time imprecise calculations, where the output results can be easily evaluated with respect to the inputs, and where the *backward path* can be utilized without further computation.

3 The Probability Approach

The first portion of this section will present the background necessary to construct the probability approach. Operation rules for imprecise calculations are then derived, followed by the interpretation scheme for carrying out these calculations. Analytical and numerical applications of the method conclude this section.

The probability approach can be developed on either of two interpretations: the "classical" relative frequency of occurrences, or ("Bayesian") probability logic as a measure of plausibility of propositions [9, 16]. However, the calculus is the same for both interpretations. Here we use the calculus of probability and introduce another interpretation of its meaning: that of *preference*. For convenience we will continue to use the term "probability" although we emphasize that we are not using it with either of its usual meanings, but are simply using its calculus. This is done to compare the use of the probability calculus with the fuzzy calculus for representing and manipulating design imprecision. Naturally probability, with its usual interpretation, would continue to be used to represent stochastic uncertainty.

Only three axioms are required in the process of constructing the calculus for probability logic [9]:

$$P(a \mid b) \geq 0, \qquad \text{and} \qquad P(a \mid a) = 1, \tag{4}$$

$$P(a \mid b) + P(\tilde{a} \mid b) = 1, \tag{5}$$

$$P(a, b \mid c) = P(a \mid b, c) \cdot P(b \mid c). \tag{6}$$

The first of these axioms simply states the conventions that the probability of a given information b must be nonnegative, and the probability of a given itself is unity. The second axiom represents a statement giving the probability of the negation of a in terms of the probability of a, under the same hypothesis b, *i.e.*, the probability of a given b summed with the probability of the contradiction of a on the same information must equal unity. Finally, Axiom 6 is the product rule giving the probability of a and b under the hypothesis c in terms of more elementary probabilities, or equivalently "*the probability of the joint assertion of two propositions on any data c is the product of the probability of one of them on data c and that of the other on the first and c.*"[16] The form of the axioms given in Equations 5 and 6 is partly conventional and partly requirements for internal consistency of the calculus [9]. Axioms 5 and 6 imply that

$$P(a \textbf{ or } b \mid c) = P(a \mid c) + P(b \mid c) - P(a, b \mid c), \tag{7}$$

which we call the addition rule.

3.1 Probability Function Operations

To construct the analogous form of the fuzzy calculus method for imprecise calculations, we must first develop the rules for calculation operations. For the sake of brevity, we will only present the rules for the binary operations of addition, subtraction, multiplication, and division, along with the unitary operations for the sine and cosine functions. Appendix A presents the detailed derivations of these operations given two independent "input" parameters $\hat{u_1}$ and $\hat{u_2}$ and "output" parameter $\hat{z}$:

$$\hat{z} = f(\hat{u_1}, \hat{u_2}),$$

where the function f is made up of any combination of the six possible operations listed. The resulting probability density functions (*pdf*s) for the output parameter $\hat{z}$ are summarized below:

$$p_{add}(z \mid I) = \int_{-\infty}^{\infty} p_{\hat{u_1}}(z - u_2)\, p_{\hat{u_2}}(u_2)\, du_2. \tag{8}$$

$$p_{sub}(z \mid I) = \int_{-\infty}^{\infty} p_{\hat{u_1}}(z + u_2)\, p_{\hat{u_2}}(u_2)\, du_2, \tag{9}$$

$$p_{mul}(z \mid I) = \int_{-\infty}^{\infty} \frac{1}{u_2} p_{\hat{u_1}}(\frac{z}{u_2})\, p_{\hat{u_2}}(u_2)\, du_2, \tag{10}$$

$$p_{div}(z \mid I) = \int_{-\infty}^{\infty} u_2 \cdot p_{\hat{u_1}}(z \cdot u_2)\, p_{\hat{u_2}}(u_2)\, du_2, \tag{11}$$

$$p_{sin}(z \mid I) = \frac{1}{\sqrt{1 - z^2}} p_{\hat{u_1}}(\sin^{-1}(z)),\ (\frac{-\pi}{2} \leq \sin^{-1}(z) \leq \frac{\pi}{2}), \tag{12}$$

$$p_{cos}(z \mid I) = \frac{1}{\sqrt{1-z^2}} p_{\hat{u}_1}(\cos^{-1}(z)),\ (0 \leq \cos^{-1}(z) \leq \pi). \tag{13}$$

In the probability calculations that follow the I notation (representing the information contained in the design parameters) will be suppressed for brevity.

3.2 The Probability Interpretation

As discussed earlier, the fuzzy calculus technique represents imprecision by a range and a function defined in that range to describe the desirability or preference of using one particular value over another. The more the designer desires to use an input value, the higher its preference value in the parameter's normalized (between zero and one) preference function. A similar interpretation for the *probability* approach to representing imprecision is developed below.

The input design parameters can be described by probability density functions (*pdf*) $p_i()$ with each $p_i()$ having unit area. Figure 2 shows an example input parameter u_i at the top. The vertical axis values of the *pdf* capture the subjective imprecision or approximate character of u_i, in a way analogous to the membership for fuzzy calculus approach. $p_i(u_i)$ value(s) of maximum height correspond to the value(s) of u_i for which the designer rates as highest in terms of preference or desirability. Conversely, u_i values with $p_i(u_i)$ equal to zero are those the designer rates as least preferred or unacceptable (*e.g.*, the endpoints of the interval of confidence for u_i). A fuzzy number captures the desires of the designer by use of a function which ranges from zero to one, whereas a corresponding *pdf* for some input parameters may vary in height depending on the need to meet the unit-area condition. These two methods of representing the subjective nature of an *input* parameter's imprecision are equivalent in what follows, because the ratio of any value in the input range to the most desired input(s) is kept the same for the fuzzy and probability approaches.

To allow direct comparison with the fuzzy calculus results, the *pdf*s are therefore normalized so their maximum height is unity. Figure 2 provides an example output *pdf* from some $f(u_i)$, along with a uniformly scaled version of the output, ranging from zero to unity on the vertical axis. Such an output is constructed by normalizing each point on the curve $P(z)$ by $\max[P(z)]$. We call the scaled curve P_{output} (see Figure 2) the *relative desirability of* $\hat{z}$.

3.3 Analytical Application of the Probability Approach

Calculations with fuzzy numbers (defined as an interval of confidence and level of presumption) are performed using operation rules based on α-cuts [13, 17]. Using the interpretation scheme and operation rules defined previously, an analytical method for calculating outputs for the probability approach can be likewise developed.

Consider a general input design parameter, u_i, graphically represented by the *pdf* shown at the top of Figure 2. Here we will only consider triangular *pdf*s such that

$p_{u_i}(u_i)$ can then be defined as follows:

$$p_{u_i}(u_i) = \begin{cases} 0 & u_i < c_1 \\ \frac{2}{(c_3-c_1)(c_2-c_1)}(u_i - c_1) & c_1 \leq u_i \leq c_2 \\ \frac{2}{(c_3-c_1)(c_2-c_3)}(u_i - c_3) & c_2 \leq u_i \leq c_3 \\ 0 & u_i > c_3 \end{cases} \tag{14}$$

where

$$(c_3 - c_1) \;=\; \text{input range} \;\in \Re.$$

For a given functional requirement expression $\hat{z} = f(u_i)$, $i = 1, \ldots, N$ and known *pdf*s for $p_{u_1}(u_1), \ldots, p_{u_N}(u_N)$, we end up with $p(z)$ equal to a multiple integral expression of order $(N-1)$. In order to deal with the discontinuity in the triangular input and the intervals for which the integrals will be applied, a Heaviside function will be defined for convenience:

$$\mathcal{H}(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0. \end{cases} \tag{15}$$

$\mathcal{H}(x)$ is defined to be zero for $x = 0$ in order to properly handle the limits of integration on the integral terms for $p(z)$. The *pdf* in Figure 2 and Equation 14 can now be redefined in terms of the Heaviside function:

$$\begin{aligned} p_{u_i}(u_i) \;=\; & [\mathcal{H}(u_i - c_1) - \mathcal{H}(u_i - c_2)] \cdot D_1 \cdot (u_i - c_1) \\ & + [\mathcal{H}(u_i - c_2) - \mathcal{H}(u_i - c_3)] \cdot D_2 \cdot (u_i - c_3), \end{aligned} \tag{16}$$

where

$$D_1 = \frac{2}{(c_3 - c_1)(c_2 - c_1)} \quad \text{and}$$

$$D_2 = \frac{2}{(c_3 - c_1)(c_2 - c_3)}$$

The general form of the analytical imprecise calculation for the probability approach is given by:

$$p(z) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\ldots)\, du_{N-1} \cdots du_1, \tag{17}$$

where the term $(\ldots)$ is made up of the product of *pdf*s in the form of Equation 16. Substitution of Equation 16 for the *pdf*s in the term $(\ldots)$ will reduce Equation 17 to

$$p(z) = \sum_j [G_k(e_j(c_l, z)) - G_k(f_j(c_l, z))] \cdot \mathcal{H}(e_j(c_l, z) - f_j(c_l, z)), \tag{18}$$

where

$$\begin{aligned}
G_k(x) &= \int^x g_k(\xi)\, d\xi, \\
g_k(x) &= \text{a product of } D_1(u_i - c_1) \text{ with itself, with } D_2(u_i - c_3), \\
&\quad\ \text{or } D_2(u_i - c_3) \text{ with itself}, \\
e_j(c_l, z), & \\
f_j(c_l, z) &= \text{functions of } c_l, \text{ and } z \text{ depending on the } j^{th} \text{ operation} \\
&\quad\ \text{and limits of integration}, \\
l &= 1, \ldots, 3, \\
e_j() &\neq f_j().
\end{aligned}$$

3.4 Numerical Application of the Probability Approach

As discussed earlier, the fuzzy calculus method for analytically calculating imprecise performance parameters is impractical for computer-assisted design applications, due to algebraic complexity. A discrete numerical method is necessary to satisfy computational requirements [12, 25].

When considering the analytical application of the *probability* approach, a similar combinatorial problem arises for an increasing number of design parameters due to the resulting multiple integrations. A numerical scheme for calculating the imprecise output as expressed by Equation 17 must therefore be developed. Because the calculation rule given by Equation 17 depends on the output variable of interest (z in this case), the output range must be combinatorially determined through interval analysis and then discretized for a numerical approximation. Implementing such a discretized output range, a numerical algorithm for the probability approach can be used as follows:

1. Assuming the general case $\hat{z} = f(u_i)$, determine the upper and lower bound on $\hat{z}$, denoted by z_u and z_l, through interval analysis.
2. Discretize the output parameter $\hat{z}$: $\hat{z} = m\Delta z + z_l$, where $m = 0, 1, \ldots, \frac{(z_u - z_l)}{\Delta z}$.
3. For each integration variable u_i with corresponding triangular input function $p_{u_i}(u_i)$ (Figure 2), discretize u_i: $u_i = n_i \Delta u_i + c_1$.
4. Replace the integral(s) in Equation 17 with discretized summation(s) such that

$$p(z) = p(m\Delta z + z_l) = \Delta u_1 \cdots \Delta u_{N-1} \sum_{u_1} \cdots \sum_{u_{N-1}} (\ldots) \tag{19}$$

For N design parameters and corresponding discretizations, the complexity of the numerical algorithm is of order

$$H \sim 2^{N-1} \cdot \kappa_1 \cdot \frac{(c_3 - c1)^{N-1}}{\Delta u_1 \cdots \Delta u_{N-1}} \cdot \frac{(z_u - z_l)}{\Delta z} \cdot (N-1) \cdot \kappa_2$$

where H is the number of operations, κ_1 equals the number of multiplications and divisions in $f(u_i)$, and κ_2 equals the number of multiplications and divisions in the $(\ldots)$ term of Equation 19. If all input parameter ranges as well as the output range are discretized into M intervals, the complexity becomes:

$$H \sim (N-1) \cdot M^N \cdot \kappa_1 \cdot \kappa_2, \tag{20}$$

which can be compared to the complexity of the fuzzy calculus FWA algorithm shown in Equation 3.

4 Examples

This section will present a number of examples of imprecise calculations using the probability approach along with its analytical and numerical applications, described in Section 3. The complexity of the examples will be a progression from simple addition to calculations which involve real-world design equations. The results of both the probability approach and fuzzy calculus method are shown graphically; however, only the mathematical development for the probability method is shown due to the thorough treatment of the fuzzy calculus case in [24]. In all fuzzy calculations, the input parameters are represented with an interval of confidence and a preference function, and the FWA algorithm is applied to obtain the output, where, for these examples, M (the number of discretized preference function points) is equal to 11.

4.1 Example 1: Analytical Addition

Given two input parameters u_1 and u_2 both in the form of the top of Figure 2, the problem is to calculate the output

$$\hat{z} = u_1 + u_2 \tag{21}$$

using the analytical method. From Equation 8, $p(z)$ is given by

$$p(z) = \int_{u_2} p_{u_1}(z - u_2)\, p_{u_2}(u_2)\, du_2.$$

Substituting p_{u_1} and p_{u_2} into Equation 16:

$$\begin{aligned} p(z) \quad = \quad & \int_{-\infty}^{\infty} \{[\mathcal{H}((z-u_2)-c_1) - \mathcal{H}((z-u_2)-c_2)] \cdot D_1 \cdot ((z-u_2)-c_1) \\ & +[\mathcal{H}((z-u_2)-c_2) - \mathcal{H}((z-u_2)-c_3)] \cdot D_2 \cdot ((z-u_2)-c_3)\} \cdot \\ & \{[\mathcal{H}(u_2-c_1) - \mathcal{H}(u_2-c_2)] \cdot D_1 \cdot (u_2-c_1) \\ & +[\mathcal{H}(u_2-c_2) - \mathcal{H}(u_2-c_3)] \cdot D_2 \cdot (c_2-u_2)\}\, du_2. \end{aligned} \tag{22}$$

Let $c_1 = 2$, $c_2 = 3$, and $c_3 = 4$. Multiplying the terms of Equation 22 and applying Equation 18, the result (after algebraic manipulation) is

$$\begin{aligned} p(z) &= \mathcal{H}(z-4)\cdot\{\frac{z^3}{6} - 2z^2 + 8z - \frac{32}{3}\} \\ &\quad -2\mathcal{H}(z-5)\cdot\{\frac{z^3}{3} - 5z^2 + 25z - \frac{125}{3}\} \\ &\quad +6\mathcal{H}(z-6)\cdot\{\frac{z^3}{6} - 3z^2 + 18z - \frac{108}{3}\} \\ &\quad -2\mathcal{H}(z-7)\cdot\{\frac{z^3}{3} - 7z^2 + 49z - \frac{343}{3}\} \\ &\quad +\mathcal{H}(z-8)\cdot\{\frac{z^3}{6} - 4z^2 + 32z - \frac{256}{3}\}. \end{aligned} \tag{23}$$

Considering z in the output interval $[4, 8]$, the *pdf* $p(z)$ can be constructed as shown in Figure 3. The corresponding $P_{output}(z)$ curve and the fuzzy result for z can also be determined, Figure 3.

Comparing the results, we find that the probability approach output approaches a normal distribution (recall the *central limit theorem*), while the fuzzy approach results in a triangular function. The peaks of $P_{output}(z)$ and the fuzzy output both fall at the sum of the most preferred inputs, $z_{peak} = 6.0$ (Figure 3). The probability and fuzzy output curves are not of identical form; however, very similar results occur for the output ranges where the vertical axis values (Figure 3) fall between 0.7 and 1.0.

4.2 Example 2: Numerical Addition

Given the same problem as Example 1 (Equation 21), this example will illustrate the numerical application of the probability approach. The upper and lower bound on z are:

$$\begin{aligned} z_l &= 4.0, \\ z_u &= 8.0. \end{aligned}$$

The discretizations for $\hat{z}$ and u_2 can be specified as

$$\begin{aligned} z &= m\Delta z + 4.0, \\ u_2 &= n_2\Delta u_2 + 4.0. \end{aligned}$$

Substituting into Equation 19, we have

$$p(z) = \Delta u_2 \sum_{n_2} p_{u_1}(z - u_2)\, p_{u_2}(u_2). \tag{24}$$

Letting $m, n_2 = 0, 1, \ldots, 1000$ such that $\Delta z = 0.004$ and $\Delta u_2 = 0.004$, the calculated result of Equation 24 can be determined as in Figure 4.

When comparing the fuzzy result of Figure 4 with the probability approach, the same statements can be made as for Example 1, where $z_{peak} = 6.0$. Notice from the figure that the numerical scheme, when compared with the analytical results, is accurate within the small numerical error.

4.3 Example 3: Linear Equation

In this case, the governing equation is given by

$$\hat{z} = u_1 \cdot u_2 + u_3 \tag{25}$$

where the top of Figure 2 represents u_i, $i = 1, 2, 3$ for $c_1 = 2$, $c_2 = 3$, and $c_3 = 4$. The numerical approach will be used here. The bounds on z are then:

$$\begin{aligned} z_l &= 6.0, \\ z_u &= 20.0. \end{aligned}$$

Discretizations for z, u_2, and u_3 can be expressed as

$$\begin{aligned} z &= m\Delta z + 6.0, \\ u_2 &= n_2 \Delta u_2 + 2.0, \\ u_3 &= n_3 \Delta u_3 + 2.0. \end{aligned}$$

Equation 19 becomes

$$p(z) = \Delta u_2 \cdot \Delta u_3 \sum_{n_3} \sum_{n_2} \frac{1}{u_2} p_{u_1}(\frac{z - u_3}{u_2})\, p_{u_2}(u_2) p_{u_3}(u_3). \tag{26}$$

For discretization $n_2, n_3 = 0, 1, \ldots, 100$ and $m = 0, 1, \ldots, 1000$, $\Delta u_2 = 0.02$, $\Delta u_3 = 0.02$ and $\Delta z = 0.014$. Substituting these values into Equation 26, the calculated result of Equation 25 can be found as in Figure 5, along with the fuzzy sets result.

In contrast to the addition operation examples (Example 1 and 2), the probability and fuzzy output curves are not symmetric about the peak. Furthermore, due to the nonlinear nature of Equation 26 (the division by u_2 is introduced by the multiplication operation), the peak of $P_{output}(z)$ does not fall at the linear combination of the most preferred inputs, but instead at $z_{peak} = 11.6$. The fuzzy result's peak, on the other hand, does correspond to such a combination, $z_{peak} = 12.0$. In general, the curves have very little similarity because of the shifted peaks as well as the collapsed range of the $P_{output}(z)$ curve compared to the fuzzy output.

4.4 Example 4: Trigonometric Operations

This example will consider simple trigonometric operations on a single parameter u_1. Only the output parameter, z, requires discretization due to the closed form of Equations 12 and 13.

4.4.1 Sine Operation

The equation of interest in this case is

$$\hat{z}_{sin} = \sin u_1, \tag{27}$$

where the input parameter u_1 is a triangular function with the peak at $u_{1,peak} = \frac{\pi}{6}$ and interval range of $u_1 \in [0.0, \frac{\pi}{3}]$. From Equation 12, the output parameter's *pdf* can be calculated such that

$$p(z_{sin}) = \frac{1}{\sqrt{1 - z_{sin}^2}} \, p_{u_1}(\sin^{-1}(z_{sin})). \tag{28}$$

Considering the output range $z_{sin} \in [0.0, \frac{\sqrt{3}}{2}]$, the result of the calculation $p(z_{sin})$ along with $P_{output}(z_{sin})$ are shown in Figure 6.

4.4.2 Cosine Operation

Using the same input parameter u_1 as for the sine operation, this case will calculate the cosine output:

$$z_{cos} = \cos u_1, \tag{29}$$

where the resulting *pdf* can be determined from

$$p(z_{cos}) = \frac{1}{\sqrt{1 - z_{cos}^2}} \, p_{u_1}(\cos^{-1}(z_{cos})). \tag{30}$$

Figure 6 shows the output curves $p(z_{cos})$ and $P_{output}(z_{cos})$, where $z_{cos} \in [0.5, 1.0]$.

The limits on the ranges of the input angles for the trigonometric operations used in this example were chosen to produce monotonic outputs. The sine output functions are found to be very similar for the probability approach and the fuzzy approach ($u_1 \in [0.0, \frac{\pi}{3}]$). The cosine function for the same input, however, produces dissimilar results as the input parameter approaches a value for which the cosine function has zero slope. Instead of $P_{output}(z)$ approaching zero for the right extreme input (Figure 6), as in the other examples, the output values are very high relative to the peak. Notice that both trigonometric operations resulted in peak values corresponding to the most preferred input, *i.e.*, for $u_1 = \frac{\pi}{6}$, $z_{sin,peak} = 0.5$ and $z_{cos,peak} = \frac{\sqrt{3}}{2}$.

4.5 Example 5: Beam Shear Stress

The problem here is to design a horizontal beam which will not fail when subjected to a vertical load distributed along its length. The configuration under consideration for this example is a simply supported beam with a pin connection on the left, a roller connection on the right, a rectangular cross-section and a uniformly distributed vertical load. Given design parameters of beam length L, width b, height h, and applied load w, one important performance parameter is the maximum shear stress, τ:

$$\tau = \frac{3\,w\,L}{4\,b\,h}. \tag{31}$$

Other performance parameters of interest, which we do not consider here, might be: mid-point deflection, maximum bending stress, etc. Table 1 lists the representative data (left-extreme, right-extreme, and peak values) for the triangular design inputs, where we define $\hat{w} = \frac{3}{4}w$ so that Equation 31 becomes

$$\tau = \frac{\hat{w}\, L}{b\, h}. \tag{32}$$

Applying Equations 10 and 11, $p(\tau)$ is given by

$$p(\tau) = \int_h \int_b \int_L \left(\frac{h \cdot b}{L}\right) p_{\hat{w}} \left(\frac{\tau \cdot h \cdot b}{L}\right) p_L(L) p_b(b) p_h(h)\; dL\, db\, dh. \tag{33}$$

The numerical solution to Equation 33 can be formulated as follows:

1. Combinatorially determine upper and lower bounds for τ.

$$\begin{aligned} \tau_l &= 0.9 \text{ MPa}, \\ \tau_u &= 202.5 \text{ MPa}. \end{aligned}$$

   Note the large range (typical in preliminary engineering design) for the performance parameter τ ($> 10^2$) that results from reasonable ranges for the 5 design parameters (Table 1). This demonstrates the need for a computationally efficient method for this stage of the design process.

2. Discretize the output range for τ and the input parameters L, b, and h:

$$\begin{aligned} \tau &= m\Delta\tau + 0.9, \\ L &= n_1 \Delta L + 3.0, \\ b &= n_2 \Delta b + 0.1, \\ h &= n_3 \Delta h + 0.1. \end{aligned}$$

3. Express Equation 33 in the numerical application form:

$$p(\tau) = \Delta L \cdot \Delta b \cdot \Delta h \sum_{n_3} \sum_{n_2} \sum_{n_1} \left(\frac{h \cdot b}{L}\right) p_{\hat{w}} \left(\frac{\tau \cdot h \cdot b}{L}\right) p_L(L)\, p_b(b)\, p_h(h). \tag{34}$$

4. Letting $n_i = 0, 1, \ldots, 50$, $(i = 1, 2, 3)$ and $m = 0, 1, \ldots, 1000$ such that $\Delta L = 0.12$, $\Delta b = \Delta h = 0.008$, and $\Delta\tau = 0.2016$, calculate $p(\tau)$ from Equation 34. The result of such a calculation is shown in Figure 7.

Again we see that the peak of the curve produced by the probability calculus does not occur at the shear stress corresponding to the most preferred design parameter values.

4.6 Example 6: Torque for a Drum Brake

The motivation for this example is the design of a braking system for a vehicle which will adequately stop the vehicle for a certain range of speeds. Given that a drum brake configuration is under consideration for this design, one important performance parameter is the torque $\mathcal{T}$ [21]:

$$\mathcal{T} = \mu\, p_a\, b\, r^2\,(\cos\theta_1 \,-\, \cos\theta_2), \tag{35}$$

where

$$\begin{aligned}
b &= \textit{face width of frictional material},\\
p_a &= \textit{maximum operating pressure of material},\\
r &= \textit{inner drum brake radius},\\
\mu &= \textit{coefficient of friction},\\
\theta_1 &= \textit{angle to beginning of frictional material},\\
\theta_2 &= \textit{angle to end of frictional material}.
\end{aligned}$$

The representative data for the torque input parameters (b, p_a, r, μ, θ_1, and θ_2) can be found in Table 2. Applying Equations 9, 10, and 13, the numerical scheme for determining $\mathcal{T}$ is as follows:

1. Determine the bounds for $\mathcal{T}$:

$$\begin{aligned}
\mathcal{T}_l &= 0.00375\,(\text{kN-m}),\\
\mathcal{T}_u &= 5.3181\,(\text{kN-m}).
\end{aligned}$$

   Note again the large range for the performance parameter $\mathcal{T}$ ($> 10^3$) that results from reasonable ranges for the 6 design parameters (Table 2).

2. Write the analytical form of the expression for $p(\mathcal{T})$:

$$\begin{aligned}
p(\mathcal{T}) &= \int_\mu \int_{p_a} \int_b \int_{u_3} \int_{u_2} \left(\frac{1}{u_3 \cdot \mu \cdot p_a \cdot b}\right) p_\mu(\mu) p_{p_a}(p_a) p_b(b)\\
&\quad p_{u_4}\left(\frac{\mathcal{T}}{u_3 \cdot \mu \cdot p_a \cdot b}\right)\left(\frac{1}{\sqrt{1-(u_3+u_2)^2}}\right) p_{\theta_1}(\cos^{-1}(u_3+u_2))\\
&\quad \left(\frac{1}{\sqrt{1-u_2^2}}\right) p_{\theta_2}(\cos^{-1}(u_2))\, du_2\, du_3\;db\, dp_a\, d\mu,
\end{aligned} \tag{36}$$

   where

$$\begin{aligned}
u_1 &= \cos(\theta_1),\\
u_2 &= \cos(\theta_2),\\
u_3 &= u_1 - u_2,\\
u_4 &= r^2, \text{ and}\\
p_{u_4} &= \frac{p_r(r)}{2 \cdot r}.
\end{aligned}$$

3. Discretize the input and output ranges:

$$
\begin{aligned}
\mathcal{T} &= m\Delta\mathcal{T} + \mathcal{T}_l, \\
u_3 &= n_1\Delta u_3 + 0.5, \\
\mu &= n_2\Delta\mu + 0.1, \\
p_a &= n_3\Delta p_a + 300.0, \\
b &= n_4\Delta b + 0.025, \\
u_2 &= n_6\Delta u_2 - \frac{\sqrt{3}}{2}.
\end{aligned}
$$

4. Transform Equation 36 into the form of Equation 19.

5. Letting $n_i = 0, 1, \ldots, 10$ and $m = 0, 1, \ldots, 100$, the result of the numerical calculation can be determined as shown in Figure 8.

Examples 5 and 6 consist of calculations with real-world design equations, building from the operations carried out in the previous examples. Figures 7 and 8 show that the operations that introduce nonlinearities into the probability calculus shift the peak of P_{output} when compared with the fuzzy result. The height of the probability preference function versus the fuzzy preference function also varies greatly in the output horizontal range. When compared with the previous results for Examples 1 through 4, this effect is much more dramatic for the shear stress and brake torque due to the increased number of operations.

DPs u_i (units)	$p_{u_i}(u_i)_{left} = 0$	$p_{u_i}(u_i) = 1$	$p_{u_i}(u_i)_{right} = 0$
L (m)	3.0	6.0	9.0
b (m)	0.10	0.30	0.50
h (m)	0.10	0.30	0.50
$\hat{w}$ (kN/m)	75.0	150.0	225.0

Table 1: Beam Example: Design Parameter Data.

DPs u_i (units)	$p_{u_i}(u_i)_{left} = 0$	$p_{u_i}(u_i) = 1$	$p_{u_i}(u_i)_{right} = 0$
b (m)	0.025	0.050	0.075
μ	0.10	0.30	0.50
p_a (kPa)	300.0	1100.0	1900.0
r (m)	0.10	0.15	0.20
θ_1 (deg.)	0.0	30.0	60.0
θ_2 (deg.)	90.0	120.0	150.0

Table 2: Brake Example: Design Parameter Data.

5 Discussion

Section 4 presents several examples of the probability calculus approach applied to equation calculations with *imprecise* input parameters. This section will compare the results of these examples with the fuzzy calculus approach.

The output parameters from the example calculations in Section 4 are made up of continuous *pdf*s, which must have the following two properties:

1. $p(z) \geq 0$,
2. $\int_{-\infty}^{\infty} p(z) dz = 1$.

Figures 3 through 8 possess both these properties. Furthermore, in the case of Example 1 and Example 2, the *central limit theorem* applies, *i.e.*, for a large number of uncertain parameters $u_1, \ldots, u_N$, the sum of the parameters will result in an approximate Normal (Gaussian) distribution. Figures 3 and 4 illustrate that for triangular input parameters, a single addition operation produces an output similar to a Normal distribution. The triangular shape of the *design imprecision* of the input parameters was chosen to provide a consistent comparison between the two calculi. Naturally stochastic parameters would rarely have a triangular shape, but since the comparison is on how well the two calculi represent and manipulate *design imprecision*, which may commonly have a triangular shape, it is a reasonable basis for the comparison. The comparison can be generalized to more arbitrary shapes for design imprecision.

Our discussion of the fuzzy calculus approach compared to the probability approach for representing and manipulating *design imprecision*, will be three-fold: an assessment of the general character of the output (performance parameter) preference functions in the two cases; the differences in the interpretations of these curves; and the usefulness of applying either method in the design domain. We find that two primary differences occurred in the outputs of the probability approach calculations compared to the fuzzy set method: the output peak is shifted, and the output height over the range of the performance parameter is different. The differences in the axiomatic development of the two calculi, of course, are the basis for these output dissimilarities.

The fuzzy approach relies on max-min operations using the *extension principle* [31], which assures that the peak of a calculation with triangular inputs will occur at the calculated combination of the most desirable values (peaks) of the input parameters. The peak of the output from the probability approach, on the other hand, will generally be shifted, and will have similar results to the fuzzy calculus only when the calculation involves linear operations. This is not the case for nonlinear operations, because many less-likely sets of inputs for a common output may outweigh one more-likely set of inputs for another output.

The differences in the output height for the two methods can be explained similarly. The fuzzy approach once again relies on the max-min solution, which tends to broaden the output (*i.e.*, greater preference values for points that approach the extremes of the output range interval). The probability approach, however, relies on the addition rule.

For an increasing number of operations in $f(u_i)$, combinations of input parameters with values far off the input curve's peak will contribute very little to the output curve, resulting in a collapsing of the output range (*i.e.*, small output height for the extreme output range values). The result of this narrowing is that a large interval (in the "tail") of the output performance parameter has nearly equal preference values (near zero), making differentiation between values in this range difficult. If the functional requirement falls in this region, it is difficult to infer how the input design parameters must be changed (and the designer's preference reduced) in order to meet the requirement. While some small numerical differences between performance parameters preference data will exist, their significance is questionable, as the data that formed these outputs (in the preliminary design phase) was imprecise itself. Once again, the narrowing of the peak of the curve calculated with probability calculus presents difficulties for *design imprecision* data, but (of course) is quite correct for stochastic data.

The interpretation of the output results for the two methods is also different. The result of the fuzzy approach to imprecise calculations is in general a curve which peaks (preference of one (1)) at the output value corresponding to the most preferred input values, and which has an associated interval range of all possible combinations of the input parameters. The value at any point on the output preference function can be directly traced back to the combination of input parameters which resulted in the corresponding output value. Alternatively, the result of the probability approach is a curve which peaks $P_{output}(z) = 1$ at a point shifted, in general, from the output value corresponding to the most preferred input values, and which has the same output range as the fuzzy calculus method. (If the usual interpretation of probability was being used, the peak would occur at the most likely output value). Because of the addition rule for probability, input values can not be practically traced back from a given output on the curve $P_{output}(z)$. The reason for this is that a probability calculation for a point on $P_{output}(z)$ is a combination of *all* the probabilities from inputs corresponding to a specific output z value.

The designer must be able to determine the performance parameter point (or interval) corresponding to the most preferred inputs in order to compare a performance parameter to its functional requirement. If the input design parameters combine, as in the case of probability calculations, he or she will only be able to determine a relative measure of how many different sets of inputs (of high preference) contribute to an output performance parameter value. This does not tell the designer how one configuration (*e.g.*, one set of input design parameter values) compares with a functional requirement. A design should not be determined by how many combinations of design parameters will result in a performance parameter output that satisfies the functional requirement, (as would be the case if probability calculus was used to manipulate imprecision), but instead should answer the question of "what is the sacrifice in preference required to satisfy a functional requirement for a given configuration," or "how well will one configuration meet the functional requirement?" The fuzzy calculus provides this capability.

The use of probability to represent *design imprecision* has an additional difficulty beyond the ones described above. Probability calculations with large numbers of uncertain input parameters (*e.g.*, greater than 10) will be slow and computational expensive (Equation 20), in comparison to fuzzy calculations (Equation 3). We may wish to use probability calculations for including the effects of stochastic uncertainty in parallel with the fuzzy calculus approach to design imprecision, as long as the number of stochastically uncertain parameters is small.

This paper has concentrated on a comparison of the use of two different calculi (fuzzy and probability) to represent and manipulate *design imprecision*. In a typical engineering design, stochastic uncertainty, design imprecision, and (perhaps) possibilistic uncertainty will all be present. We have introduced a technique elsewhere [26, 27], called *Extended Hybrid Numbers* for carrying all three uncertainties through design calculations, while keeping them distinct.

6 Conclusions

Using the analysis of the differences in output curves and interpretation as discussed above, we can determine the applicability and usefulness of the fuzzy approach and probability approach for the representing and manipulating imprecise parameters in the design domain. The fuzzy approach is a technique which presents more information to a designer than conventional single valued or interval analysis, by indicating the relative importance of input parameters as well as providing a method for comparing different solution alternatives. The probability approach can satisfy our first objective in that the subjectivity of the designer can be represented and manipulated. However, the performance parameter results are in a form which makes the evaluation of the relative importance of inputs more difficult, due to the narrowing of the output peak common to probability calculations (as shown in the examples). The shifting of the peak of the output from the probability calculations, away from the combination of the most preferred inputs, also reduces their usefulness. Additionally, the ability to trace an output value back to a set of inputs that produced it is absent in probability calculations with imprecise parameters. Finally, probability calculations are far more computationally complex than fuzzy calculations. All of the above contribute to making fuzzy calculations on design imprecision more applicable and useful for preliminary engineering design.

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A Appendix: Derivation of the Probability Operations

The following section contains the derivation of the probability approach operations summarized in Section 3.1.

In general, let $\hat{u_1}$ and $\hat{u_2}$ be two independent uncertain or imprecise "input" parameters, and let I be the proposition stating that their imprecision or uncertainty is assumed to be described by the subjectively assigned probability density functions (*pdf*s) $p_{\hat{u_1}}(u_1)$ and $p_{\hat{u_2}}(u_2)$, that is,

$$\begin{aligned} P(\hat{u_1} \leq u_1 \mid I) &= \int_{-\infty}^{u_1} p_{\hat{u_1}}(\xi) d\xi, \\ &\text{and} \\ P(\hat{u_2} \leq u_2 \mid I) &= \int_{-\infty}^{u_2} p_{\hat{u_2}}(\eta) d\eta. \end{aligned} \tag{37}$$

Note that for a finite range of possible values, $p_{\hat{u_1}}(u_1)$ and $p_{\hat{u_2}}(u_2)$ would be set to zero for u_1 and u_2 outside their respective input ranges.

Let $\hat{z}$ be the uncertain or imprecise "output" parameter, given by:

$$\hat{z} = f(\hat{u_1}, \hat{u_2}).$$

We wish to determine the resulting *pdf* for $\hat{z}$. From the axioms in Equations 4 to 6:

$$\begin{aligned} P(\hat{z} \leq z \mid I) &= P(\hat{z} \leq z, \hat{u_2} \in (-\infty, \infty) \mid I) \\ &= \sum_{n=-\infty}^{\infty} P(\hat{z} \leq z, \hat{u_2} \in [u_{2,n}, u_{2,n} + \delta u_2] \mid I) \\ &= \sum_{n=-\infty}^{\infty} P(\hat{z} \leq z \mid \hat{u_2} \in [u_{2,n}, u_{2,n} + \delta u_2], I) \times P(\hat{u_2} \in [u_{2,n}, u_{2,n} + \delta u_2] \mid I) \\ &= \sum_{n=-\infty}^{\infty} P(\hat{z} \leq z \mid \hat{u_2} \in [u_{2,n}, u_{2,n} + \delta u_2], I) \cdot p_{\hat{u_2}}(u_{2,n})\, \delta u_2, \end{aligned}$$

where the real line is divided into equal intervals δu_2 by the points $u_{2,n}, n = 0, \pm 1, \pm 2, \ldots, \pm\infty$. Taking the limit as δu_2 tends to zero, we get:

$$P(\hat{z} \leq z \mid I) = \int_{-\infty}^{\infty} P(\hat{z} \leq z \mid \hat{u_2} = u_2, I) p_{\hat{u_2}}(u_2) du_2. \tag{38}$$

We apply this result to each of the binary operations.

A.1 Addition

Here $\hat{z} = \hat{u_1} + \hat{u_2}$; so from Equation 37:

$$\begin{aligned} P(\hat{z} \leq z \mid \hat{u_2} = u_2, I) &= P(\hat{u_1} + \hat{u_2} \leq z \mid \hat{u_2} = u_2, I) \\ &= P(\hat{u_1} \leq z - u_2 \mid I) \\ &= \int_{-\infty}^{z-u_2} p_{\hat{u_1}}(u_1) du_1. \end{aligned}$$

Therefore,

$$\frac{d}{dz}P(\hat{z} \leq z \mid \hat{u_2} = u_2, I) = p_{\hat{u_1}}(z - u_2),$$

and

$$\begin{aligned} p_{add}(z \mid I) &= \frac{d}{dz}P(\hat{z} \leq z \mid I) \\ &= \int_{-\infty}^{\infty} p_{\hat{u_1}}(z - u_2)p_{\hat{u_2}}(u_2)du_2. \end{aligned} \tag{39}$$

A.2 Subtraction

Here $\hat{z} = \hat{u_1} - \hat{u_2}$, and, as above, we get:

$$p_{sub}(z \mid I) = \int_{-\infty}^{\infty} p_{\hat{u_1}}(z + u_2)p_{\hat{u_2}}(u_2)du_2. \tag{40}$$

A.3 Multiplication

Here $\hat{z} = \hat{u_1} \cdot \hat{u_2}$; so, assuming first that $u_2 \neq 0$:

$$\begin{aligned} P(\hat{z} \leq z \mid \hat{u_2} = u_2, I) &= P(\hat{u_1} \cdot \hat{u_2} \leq z \mid \hat{u_2} = u_2, I) \\ &= P(\hat{u_1} \cdot u_2 \leq z \mid I) \\ &= P(\hat{u_1} \leq \frac{z}{u_2} \mid I) \\ &= \int_{-\infty}^{\frac{z}{u_2}} p_{\hat{u_1}}(u_1)du_1. \end{aligned}$$

Therefore,

$$\frac{d}{dz}P(\hat{z} \leq z \mid \hat{u_2} = u_2, I) = \frac{1}{u_2}p_{\hat{u_1}}(\frac{z}{u_2}).$$

If $u_2 = 0$ is a possible value of $\hat{u_2}$, we note that this function approaches zero as u_2 tends to zero since $p_{\hat{u_1}}(u_1)$ must be integrable over $(-\infty, \infty)$. Thus, regardless of whether $u_2 = 0$ is possible or not:

$$p_{mul}(z \mid I) = \int_{-\infty}^{\infty} \frac{1}{u_2}p_{\hat{u_1}}(\frac{z}{u_2})p_{\hat{u_2}}(u_2)du_2. \tag{41}$$

A.4 Division

Here $\hat{z} = \frac{\hat{u_1}}{\hat{u_2}}$.

$$\begin{aligned} P(\hat{z} \leq z \mid \hat{u_2} = u_2, I) &= P(\frac{\hat{u_1}}{\hat{u_2}} \leq z \mid \hat{u_2} = u_2, I) \\ &= P(\hat{u_1} \leq u_2 \cdot z \mid I) \\ &= \int_{-\infty}^{u_2 \cdot z} p_{\hat{u_1}}(u_1)du_1. \end{aligned}$$

Therefore,

$$\frac{d}{dz}P(\hat{z} \leq z \mid \hat{u_2} = u_2, I) = u_2 p_{\hat{u_1}}(u_2 \cdot z).$$

Again, there is no complication if $u_2 = 0$ is a possible value, and:

$$p_{div}(z \mid I) = \int_{-\infty}^{\infty} u_2 p_{\hat{u_1}}(u_2 \cdot z) p_{\hat{u_2}}(u_2) du_2. \tag{42}$$

A.5 Sine

Here $\hat{z} = \sin \hat{u_1}$; so $\hat{z} \in [a, b] \subseteq [-1, 1]$, and:

$$\begin{aligned} P(\hat{z} \leq z \mid I) &= P(\sin \hat{u_1} \leq z \mid I) \\ &= P(\hat{u_1} \in \bigcup_{n=-\infty}^{\infty} [(2n-1)\pi - \sin^{-1} z, 2n\pi + \sin^{-1} z] \mid I) \\ &= \sum_{n=-\infty}^{\infty} P(\hat{u_1} \in [(2n-1)\pi - \sin^{-1} z, 2n\pi + \sin^{-1} z] \mid I), \end{aligned}$$

where $\sin^{-1} z$ is the unique value $u_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin u_1 = z$. Therefore,

$$P(\hat{z} \leq z \mid I) = \sum_{-\infty}^{\infty} [P(\hat{u_1} \leq 2n\pi + \sin^{-1} z \mid I) - P(\hat{u_1} \leq (2n-1)\pi - \sin^{-1} z \mid I)],$$

which implies:

$$\begin{aligned} p_{sin}(z \mid I) &= \frac{d}{dz} P(\hat{z} \leq z \mid I) \\ &= \frac{1}{\sqrt{1-z^2}} \sum_{-\infty}^{\infty} [p_{\hat{u_1}}(2n\pi + \sin^{-1} z) + p_{\hat{u_1}}((2n-1)\pi - \sin^{-1} u_1)]. \end{aligned} \tag{43}$$

If the possible values of $\hat{u_1}$ lie in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$, then $p_{\hat{u_1}}(u_1) = 0$ for u_1 outside this range, and:

$$p_{sin}(z \mid I) = \frac{1}{\sqrt{1-z^2}} p_{\hat{u_1}}(\sin^{-1} z). \tag{44}$$

A.6 Cosine

Here $\hat{z} = \cos \hat{u_1}$, and:

$$\begin{aligned} P(\hat{z} \leq z \mid I) &= P(\cos \hat{u_1} \leq z \mid I) \\ &= P(\hat{u_1} \in \bigcup_{n=-\infty}^{\infty} [2n\pi + \cos^{-1} z, 2(n+1)\pi - \cos^{-1} z] \mid I) \\ &= \sum_{n=-\infty}^{\infty} P(\hat{u_1} \in [2n\pi + \cos^{-1} z, 2(n+1)\pi - \cos^{-1} z] \mid I), \\ &= \sum_{-\infty}^{\infty} [P(\hat{u_1} \leq 2(n+1)\pi - \cos^{-1} z \mid I) - P(\hat{u_1} \leq 2n\pi + \cos^{-1} z \mid I)], \end{aligned}$$

where $\cos^{-1} z$ is the unique value $u_1 \in [0, \pi]$ such that $\cos u_1 = z$. Therefore,

$$p_{cos}(z \mid I) = \frac{1}{\sqrt{1-z^2}} \sum_{-\infty}^{\infty} [p_{\hat{u}_1}(2(n+1)\pi - \cos^{-1} z) + p_{\hat{u}_1}(2n\pi + \cos^{-1} u_1)]. \tag{45}$$

If the possible values of $\hat{u}_1$ lie in the range $(0, \pi)$, then $p_{\hat{u}_1}(u_1) = 0$ for u_1 outside this range, and:

$$p_{cos}(z \mid I) = \frac{1}{\sqrt{1-z^2}} p_{\hat{u}_1}(\cos^{-1} z). \tag{46}$$

B Powers of Uncertain Parameters

The brake torque example in Section 4.6 includes a DP which is raised to a power. The discussion below develops the operation rule for powers.

Consider the case where an uncertain parameter, $\hat{u}_1$, is raised to a non-zero, crisp (certain) power x, where x may either be a fraction or an integer. We wish to determine the *pdf* $p_{\hat{z}}(z)$ for $\hat{z} = \hat{u}_1{}^x$. It will be assumed here that the uncertainty for $\hat{u}_1$ is described by a subjectively assigned *pdf* $p_{\hat{u}_1}(u_1)$, and we will only consider $x \in \Re^+$ and $\hat{u}_1 \in \Re^+$ (*i.e.*, $p_{\hat{u}_1}(u_1) \equiv 0$ for $-\infty \leq u_1 \leq 0$).

From the basic definition of the cumulative distribution,

$$\begin{aligned} P(\hat{z} \leq z \mid I) &= P(\hat{u}_1{}^x \leq z \mid I) \\ &= P(\hat{u}_1 \leq z^{\frac{1}{x}} \mid I) \\ &= \int_{-\infty}^{z^{\frac{1}{x}}} p_{\hat{u}_1}(u_1)\, du_1. \end{aligned}$$

Taking the derivatives of both sides of this result and applying the chain rule:

$$\begin{aligned} \frac{d}{dz} P(\hat{z} \leq z \mid I) &= \frac{d}{dz} \int_{-\infty}^{z^{\frac{1}{x}}} p_{\hat{u}_1}(u_1)\, du_1 \\ &= p_{\hat{u}_1}(z^{\frac{1}{x}}) \frac{d}{dz}(z^{\frac{1}{x}}) \\ &= p_{\hat{u}_1}(z^{\frac{1}{x}}) \frac{1}{x} z^{(\frac{1}{x}-1)}. \end{aligned}$$

Using the basic definition for a *pdf* from Equation 37, the resulting probability density function for $\hat{z}$ may be expressed as:

$$p_{\hat{z}}(z) = \frac{p_{\hat{u}_1}(z^{\frac{1}{x}})}{x\, z^{(1-\frac{1}{x})}}. \tag{47}$$

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