

Optimization of Fast-Decodable Full-Rate STBC with Non-Vanishing Determinants

Tian Peng Ren, Yong Liang Guan, Chau Yuen, Yue Zhou and Er Yang Zhang

Abstract—Full-rate STBC (space-time block codes) with non-vanishing determinants achieve the optimal diversity-multiplexing tradeoff but incur high decoding complexity. To permit fast decoding, Sezginer, Sari and Biglieri proposed an STBC structure with special QR decomposition characteristics. In this paper, we adopt a simplified form of this fast-decodable code structure and present a new way to optimize the code analytically. We show that the signal constellation topology (such as QAM, APSK, or PSK) has a critical impact on the existence of non-vanishing determinants of the full-rate STBC. In particular, we show for the first time that, in order for APSK-STBC to achieve non-vanishing determinant, an APSK constellation topology with constellation points lying on square grid and ring radius $\sqrt{m^2 + n^2}$ (m, n integers) needs to be used. For signal constellations with vanishing determinants, we present a methodology to analytically optimize the full-rate STBC at specific constellation dimension.

I. INTRODUCTION

MULTI-INPUT multi-output (MIMO) systems can be designed to provide two types of gains: transmit diversity gain and spatial multiplexing gain [1]. The full-rate full-diversity space-time block codes (STBC) in [2]–[6] can achieve both for 2×2 MIMO systems. Recently, a fast-decodable full-rate STBC is proposed by S. Sezginer, H. Sari and E. Biglieri [5] [7]:

$$\mathbf{X}_{SSB} = \begin{bmatrix} as_1 + bs_3 & -cs_2^* - ds_4^* \\ as_2 + bs_4 & cs_1^* + ds_3^* \end{bmatrix} \quad (1)$$

where $s_i \in \mathbb{C}$ with $i = 1, \dots, 4$ are information symbols, a, b, c and $d \in \mathbb{C}$ are design coefficients and $(\cdot)^*$ denotes the complex conjugate. Due to its code structure in (1), \mathbf{X}_{SSB} has additional zero entries appearing in the upper-triangular matrix after QR decomposition of the equivalent channel matrix, thus making it fast-decodable [5] [8].

It is shown in [5] that the code structure (1) after optimizing for non-vanishing determinant can be rewritten with a single design coefficient. Base on this knowledge, in this paper we

Manuscript received August 17, 2009; revised February 23 and October 19, 2010. This research is partly supported by the International Design Center (Grant No. G3.2 & D1.3).

T. P. Ren and E. Y. Zhang are with the College of Electronic Science and Engineering, National University of Defense Technology, Changsha 410073, China (e-mail: tpren@nudt.edu.cn; eyzhang2006@hotmail.com).

Y. L. Guan is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: eyl-guan@ntu.edu.sg).

C. Yuen is with Singapore University of Technology and Design, Singapore 279623 (e-mail: yuenchau@sutd.edu.sg).

Y. Zhou is with the College of Science, National University of Defense Technology, Changsha 410073, China (e-mail: gabelozhou@gmail.com).

Digital Object Identifier ****

adopt a simplified version of the code structure (1) by setting $a = 1$, $b = r$, $c = -jr^*$ and $d = 1$ to obtain:

$$\mathbf{X} = \begin{bmatrix} s_1 + rs_3 & jr^*s_2^* - s_4^* \\ s_2 + rs_4 & -jr^*s_1^* + s_3^* \end{bmatrix} \quad (2)$$

where $j^2 = -1$ and $r \in \mathbb{C}$ is the design coefficient with $|r| = 1$. Our objective is to analytically optimize the design coefficient r in (2) to enable the full-rate STBC to achieve non-vanishing determinants. In particular, we will consider the influence of different signal constellation topologies, including rectangular quadrature amplitude modulation (QAM), amplitude-phase shift keying (APSK) and phase shift keying (PSK), on the existence of non-vanishing determinants.

The rest of this paper is organized as follows. In Section II, the methods to optimize the design coefficient in (2) for both integer-coordinate and non-integer-coordinate signal constellations are described. Comparisons of the code in (2) with other full-rate codes are shown in Section III. This paper is concluded in Section IV.

In what follows, bold lower case and upper case letters denote vectors and matrices (sets), respectively; \mathbb{R} and \mathbb{C} denote the real and the complex number fields, respectively; $(\cdot)^R$ and $(\cdot)^I$ stand for the real and imaginary parts of a complex element vector and matrix, respectively; $[\cdot]^H$ denotes the complex conjugate transpose of a matrix; $\det(\cdot)$ denotes the determinant of a square matrix.

II. OPTIMIZATION OF DESIGN COEFFICIENTS

Following [10], the diversity gain of \mathbf{X} in (2) is denoted as $\text{rank}(\Delta\mathbf{X} \cdot \Delta\mathbf{X}^H) = \text{rank}(\Delta\mathbf{X})$ [11] where the difference matrix $\Delta\mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2$, \mathbf{X}_1 and \mathbf{X}_2 are STBC matrices based on different information symbols. A full-rank $\Delta\mathbf{X}$ guarantees that $(\Delta\mathbf{X} \cdot \Delta\mathbf{X}^H)$ is full-rank, and the code \mathbf{X} in (2) will achieve full diversity. When $\Delta\mathbf{X}$ is full rank, the coding gain can be defined as

$$\begin{aligned} \text{Coding gain} &\triangleq \min_{\Delta\mathbf{X}} [\det(\Delta\mathbf{X} \cdot \Delta\mathbf{X}^H)] \\ &= \min_{\Delta\mathbf{X}} (|\det(\Delta\mathbf{X})|^2) \end{aligned} \quad (3)$$

where

$$\begin{aligned} \det(\Delta\mathbf{X}) &= -jr^*|\Delta s_1|^2 + r|\Delta s_3|^2 + \Delta s_1\Delta s_3^* - j\Delta s_1^*\Delta s_3 - \\ &\quad (jr^*|\Delta s_2|^2 - r|\Delta s_4|^2 - \Delta s_2\Delta s_4^* + j\Delta s_2^*\Delta s_4) \\ &= r(|\Delta s_3|^2 + |\Delta s_4|^2) - jr^*(|\Delta s_1|^2 + |\Delta s_2|^2) + \\ &\quad (\Delta s_1\Delta s_3^* + \Delta s_2\Delta s_4^*) - j(\Delta s_1^*\Delta s_3 + \Delta s_2^*\Delta s_4) \end{aligned}$$

and Δs_i ($i = 1, 2, 3$ and 4) are the difference symbols of s_i .

$\det(\Delta \mathbf{X})$ can be split into two parts:

$$\det(\Delta \mathbf{X}) = d_1 - d_2 \quad (4)$$

where

$$d_1 = r(|\Delta s_3|^2 + |\Delta s_4|^2) - jr^*(|\Delta s_1|^2 + |\Delta s_2|^2), \quad (5a)$$

$$\begin{aligned} d_2 &= j(\Delta s_1^* \Delta s_3 + \Delta s_2^* \Delta s_4) - (\Delta s_1 \Delta s_3^* + \Delta s_2 \Delta s_4^*) \\ &= [(\Delta s_1 \Delta s_3^* + \Delta s_2 \Delta s_4^*)^I - (\Delta s_1 \Delta s_3^* + \Delta s_2 \Delta s_4^*)^R] \\ &\quad (1 - j). \end{aligned} \quad (5b)$$

Note that d_1 is dependent on the design coefficient r , while d_2 is decided by the difference symbols only. Since d_2 is in the form of $(1 - j)$ multiplied by a real number (determined by specific values of Δs_1 to Δs_4), if d_2^R is plotted on the x -axis and d_2^I is plotted on the y -axis, d_2 lies discretely on the line $x + y = 0$, as shown in Fig. 1 and Fig. 2. Since $\min_{\Delta \mathbf{X}} [\det(\Delta \mathbf{X} \cdot \Delta \mathbf{X}^H)] = \min_{\Delta s_1 \text{ to } \Delta s_4} (|d_1 - d_2|^2)$, $d_1 \neq d_2$ is the necessary and sufficient condition for full diversity, and this can be achieved by influencing d_1 using the design coefficient r .

Let $r = u + jv$ where $u, v \in \mathbb{R}$ and $u^2 + v^2 = 1$, we have

$$\begin{aligned} d_1 &= r(|\Delta s_3|^2 + |\Delta s_4|^2) - jr^*(|\Delta s_1|^2 + |\Delta s_2|^2) \\ &= [(|\Delta s_3|^2 + |\Delta s_4|^2)u - (|\Delta s_1|^2 + |\Delta s_2|^2)v] + \\ &\quad [(|\Delta s_3|^2 + |\Delta s_4|^2)v - (|\Delta s_1|^2 + |\Delta s_2|^2)u]j. \end{aligned} \quad (6)$$

The coding gain can be analyzed in two different cases as shown below:

Case I: $|\Delta s_1|^2 + |\Delta s_2|^2 = |\Delta s_3|^2 + |\Delta s_4|^2$

In this case,

$$d_1 = [(|\Delta s_1|^2 + |\Delta s_2|^2)(u - v)](1 - j). \quad (7)$$

Similar to d_2 , d_1 lies on the line $x + y = 0$ if d_1^R is plotted on the x -axis and d_1^I is plotted on the y -axis (as illustrated in Fig 1). The discrete loci of d_1 on the line $x + y = 0$ depend not only on Δs_1 to Δs_4 , but also on u and v (the design coefficients).

Let $d_1 = \tilde{d}_1(1 - j)$ and $d_2 = \tilde{d}_2(1 - j)$, from (7) and (5b) we get

$$\tilde{d}_1 = (|\Delta s_1|^2 + |\Delta s_2|^2)(u - v) \quad (8a)$$

$$\tilde{d}_2 = (\Delta s_1 \Delta s_3^* + \Delta s_2 \Delta s_4^*)^I - (\Delta s_1 \Delta s_3^* + \Delta s_2 \Delta s_4^*)^R \quad (8b)$$

To achieve full diversity gain ($d_1 \neq d_2$), u and v must be chosen to achieve $\tilde{d}_1 \neq \tilde{d}_2$. Note that $(u - v) \in [-\sqrt{2}, \sqrt{2}]$ due to $u^2 + v^2 = 1$. Hence, in this case the **Case I** coding gain is

$$\begin{aligned} &\min_{\Delta \mathbf{X}} [\det(\Delta \mathbf{X} \cdot \Delta \mathbf{X}^H)] \\ &= \min_{\Delta s_1 \text{ to } \Delta s_4} (|d_1 - d_2|^2) \\ &= \min_{\Delta s_1 \text{ to } \Delta s_4} (2|\tilde{d}_1 - \tilde{d}_2|^2). \end{aligned} \quad (9)$$

Case II: $|\Delta s_1|^2 + |\Delta s_2|^2 \neq |\Delta s_3|^2 + |\Delta s_4|^2$

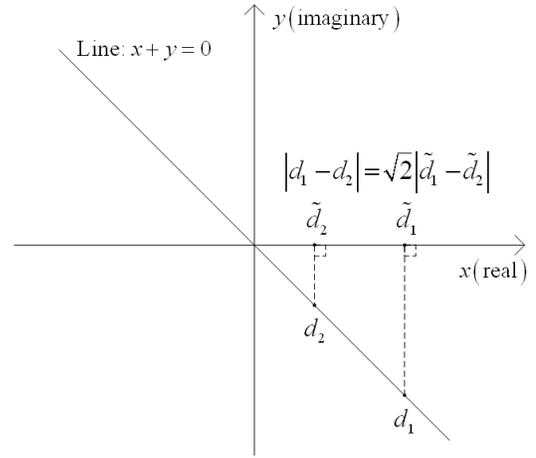


Fig. 1. d_1 , d_2 and $|d_1 - d_2|$ of **Case I** illustrated in real-imaginary axis graph.

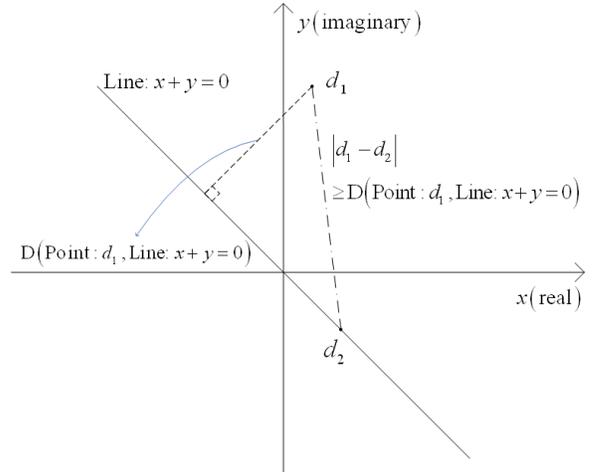


Fig. 2. d_1 , d_2 and $|d_1 - d_2|$ of **Case II** illustrated in real-imaginary axis graph.

In this case, we have

$$\begin{aligned} &d_1^R + d_1^I \\ &= (|\Delta s_3|^2 + |\Delta s_4|^2)u - (|\Delta s_1|^2 + |\Delta s_2|^2)v + \\ &\quad (|\Delta s_3|^2 + |\Delta s_4|^2)v - (|\Delta s_1|^2 + |\Delta s_2|^2)u \\ &= (|\Delta s_3|^2 + |\Delta s_4|^2 - |\Delta s_1|^2 - |\Delta s_2|^2)(u + v) \\ &= \text{a non-zero real number} \times (u + v) \end{aligned} \quad (10)$$

If $u + v \neq 0$, then $d_1^R + d_1^I \neq 0$ and d_1 will never lie on the line $x + y = 0$, as shown in Fig. 2. Since we have earlier shown that d_2 always lies on the line $x + y = 0$, it implies that $d_1 \neq d_2$, hence full diversity is always achieved by **Case II**. As shown in Fig. 2, the Euclidean distance between d_1 and d_2 can be lower bounded by the perpendicular distance between d_1 and the line $x + y = 0$ where d_2 lies on. Hence, the

Case II coding gain is lower bounded as

$$\begin{aligned}
& \min_{\Delta \mathbf{X}} [\det(\Delta \mathbf{X} \cdot \Delta \mathbf{X}^H)] \\
&= \min_{\Delta s_1 \text{ to } \Delta s_4} (|d_1 - d_2|^2) \\
&\geq \min_{\Delta s_1 \text{ to } \Delta s_4} [D^2(\text{Point} : d_1, \text{Line} : x + y = 0)] \quad (11) \\
&= \min_{\Delta s_1 \text{ to } \Delta s_4} [(|\Delta s_3|^2 + |\Delta s_4|^2 - |\Delta s_1|^2 - |\Delta s_2|^2)^2 \\
&\quad (u + v)^2 / 2]
\end{aligned}$$

where $D(\text{Point}, \text{Line})$ denotes the perpendicular distance from a point to a line.

Next, we will find the u and v that satisfy the above full diversity conditions and maximize the coding gain $\min_{\Delta \mathbf{X}} [\det(\Delta \mathbf{X} \cdot \Delta \mathbf{X}^H)]$ for the above two cases. To make the optimization process tractable, we will first find the u and v that maximize the **Case I** coding gain (9). Then we show that the **Case II** coding gain (11) substituted with the u and v obtained are larger than the maximized (9). Hence the **Case I** coding gain (9) dominates the performance of the code \mathbf{X} , and the u and v obtained by maximizing (9) will be the global optimum design coefficients.

A. Integer-Coordinate Signal Constellations

When integer-coordinate signal constellations (such as rectangular QAM) are applied, the difference symbols also have integer coordinates [2] [8], where the minimum Euclidean distance in the signal constellation is fixed at 1.

Lemma 1. When integer-coordinate signal constellations are applied, the coding gain (9) of the code \mathbf{X} in **Case I** is upper bounded by $1/2$, and the maximum value can be achieved if and only if $u - v = \pm 1/2$.

Proof: In Appendix A. ■

The following theorem establishes the non-vanishing determinant of $\Delta \mathbf{X} \cdot \Delta \mathbf{X}^H$ with integer-coordinate signal constellations.

Theorem 1. When integer-coordinate signal constellations are applied, the optimum design coefficient $r = u + jv$ to achieve full diversity and maximum non-vanishing coding gain for the code \mathbf{X} in (2) is given by:

$$u = (1 \pm \sqrt{7})/4, \quad v = (-1 \pm \sqrt{7})/4 \quad (12a)$$

$$\text{or } u = (-1 \pm \sqrt{7})/4, \quad v = (1 \pm \sqrt{7})/4. \quad (12b)$$

Proof: Let us first consider **Case I**. (12) can be obtained by combining $u - v = \pm 1/2$ from Lemma 1 and $u^2 + v^2 = 1$ (by definition).

Next, for **Case II**, since $|\Delta s_1|^2 + |\Delta s_2|^2 \neq |\Delta s_3|^2 + |\Delta s_4|^2$ and integer-coordinate signal constellations are applied, we have $||\Delta s_3|^2 + |\Delta s_4|^2 - |\Delta s_1|^2 - |\Delta s_2|^2| \geq 1$. Substituting the u and v in (12) to (11), the coding gain becomes

$$\begin{aligned}
& \min_{\Delta \mathbf{X}} [\det(\Delta \mathbf{X} \cdot \Delta \mathbf{X}^H)] \\
&\geq \min_{\Delta s_1 \text{ to } \Delta s_4} [(|\Delta s_3|^2 + |\Delta s_4|^2 - |\Delta s_1|^2 - |\Delta s_2|^2)^2 \\
&\quad (u + v)^2 / 2] \quad (13) \\
&= 7/8.
\end{aligned}$$

Comparing the **Case II** coding gain expressed in (13) with the **Case I** coding gain expressed in Lemma 1, the **Case I** coding gain is lower, hence it is the overall coding gain, and Theorem 1 is proved. ■

Remark: The method of proof in this paper, specifically **Case I** and **Case II**, are presented in a different way than the proof provided in [5]. Interestingly, however, the optimized design coefficients in both papers are found to be the same.

Application 1: Integer-coordinate APSK

APSK (amplitude-phase shift keying) is a high-order modulation scheme commonly used in SISO (single-input single-output) communications. Conventional APSK topology resembles multi-ring PSK, or circular QAM, as illustrated in Fig. 3(b) and Fig. 3(e). Compared with rectangular QAM, APSK has advantages such as lower constellation peak-to-average-power ratio (PAPR) and robustness against nonlinear distortion in SISO communications [13]. Moreover, APSK may lead to larger minimum Euclidean distance per unit average power for certain constellation dimension such as 8-APSK [14]. Hence APSK has been adopted by the DVB-S2 Standard [15].

In order for the APSK constellation with arbitrary constellation dimension to achieve non-vanishing coding gain with the code \mathbf{X} in (2), we may deduce from Theorem 1 that:

- (1) The APSK constellation points should lie on square grids and ring radius $\sqrt{m^2 + n^2}$ (m, n integers);
- (2) The design coefficient in (12) should be adopted for the code \mathbf{X} .

Two examples of the proposed APSK constellations are shown in Fig. 3(c) and Fig. 3(f). With minimum Euclidean distance fixed at 1, they lead to a non-vanishing coding gain of $1/2$ for the code \mathbf{X} (same proof as Theorem 1).

B. Non-Integer-Coordinate Signal Constellations

When non-integer-coordinate signal constellations such as M -ary phase shift keying (M -PSK) are applied, the difference symbols Δs_i ($i = 1, 2, 3$ and 4) do not have integer coordinates. This leads to a vanishing determinant for the code \mathbf{X} in (2) even when the minimum Euclidean distance is fixed at 1. The proof is straightforward, hence omitted.

Although the code \mathbf{X} in (2) with M -PSK constellations has vanishing determinant, the code can still be analytically optimized for a specific constellation size based on the mathematical framework presented earlier. The optimization methodology is described below:

Step 1 Consider **Case I**: $|\Delta s_1|^2 + |\Delta s_2|^2 = |\Delta s_3|^2 + |\Delta s_4|^2$, whose coding gain expression is shown in (9). Given a signal constellation, find out all the values of $(|\Delta s_1|^2 + |\Delta s_2|^2)$; For each value of $(|\Delta s_1|^2 + |\Delta s_2|^2)$, find out all the values of \tilde{d}_2 . Since \tilde{d}_1 is a function of $(u - v)$, the expression of $|\tilde{d}_1 - \tilde{d}_2|$ as a function of $(u - v)$ can be evaluated. Based on these expressions of $|\tilde{d}_1 - \tilde{d}_2|$ and $(u - v) \in [-\sqrt{2}, \sqrt{2}]$, obtain the maximum value of $\min_{\Delta s_1 \text{ to } \Delta s_4} |\tilde{d}_1 - \tilde{d}_2|$, and the corresponding $(u - v)$. Combining the $(u - v)$ obtained with $u^2 + v^2 = 1$, we

can obtain the corresponding u, v and the maximized coding gain;

Step 2 Next, consider **Case II**: $|\Delta s_1|^2 + |\Delta s_2|^2 \neq |\Delta s_3|^2 + |\Delta s_4|^2$. Substitute the u and v obtained in Step 1 into (11) to obtain the **Case II** coding gain. If the **Case II** coding gain is higher than that of **Case I**, then the latter is the overall coding gain by definition, and we conclude that the code \mathbf{X} in (2) with design coefficients $r = u + jv$ obtained in **Case I** achieves full diversity gain and maximum coding gain. For PSK and conventional APSK, this is found to be always true.

Application 2: 8-PSK

Applying the optimization steps described above to the code \mathbf{X} in (2) with 8-PSK constellation, the optimum design coefficients shown in (14) and the maximum coding gain of $(22572 - 15912\sqrt{2})/2401$ are obtained.

$$u = \left(11 + 6\sqrt{2} \pm \sqrt{4609 - 132\sqrt{2}} \right) / 98, \quad (14a)$$

$$v = \left(-11 - 6\sqrt{2} \pm \sqrt{4609 - 132\sqrt{2}} \right) / 98 \quad (14b)$$

or
$$u = \left(-11 - 6\sqrt{2} \pm \sqrt{4609 - 132\sqrt{2}} \right) / 98, \quad (14c)$$

$$v = \left(11 + 6\sqrt{2} \pm \sqrt{4609 - 132\sqrt{2}} \right) / 98. \quad (14d)$$

Application 3: Conventional APSK

Similarly, the optimized design coefficients and coding gains for the conventional 8-APSK shown in Fig. 3(b) and the conventional 16-APSK shown in Fig. 3(e) can be found, and are listed in Table II. They will be used later in Fig. 6.

III. SIMULATIONS AND DISCUSSIONS

In the simulations, we assume that the Rayleigh fading channel is quasi-static in the sense that the channel coefficients do not change within a codeword, and the channel state information (CSI) is perfectly known at the receiver.

A. Integer-Coordinate Signal Constellations

Firstly, we show the ML bit error rate (BER) performances of the proposed code \mathbf{X} in (2) with the optimized coefficient (12), the Golden [2], PGA [4], MTD and MCC [6] codes¹ for 2×2 MIMO systems with 4-QAM and 16-QAM in Fig. 4. The SSB code [5] is equivalent to the proposed code \mathbf{X} . The results show that the proposed code \mathbf{X} in (2) with design coefficients (12) has BER performance slightly worse than Golden code [2], comparable with PGA code [4], and better than MTD and MCC codes [6].

On the other hand, as the proposed code structure (2) is fast-decodable, it has computational complexity order M^2 [5], same as the codes in [4] [6]. Since the computational complexity order of Golden code [2] is M^4 , the small performance loss of the proposed code compared to Golden code can be viewed as a small penalty to be paid for the complexity reduction.

¹PGA, MTD and MCC denote the Paredes-Gershman-Alkhanari, Maximum Transmit Diversity and Maximum Channel Capacity codes, respectively.

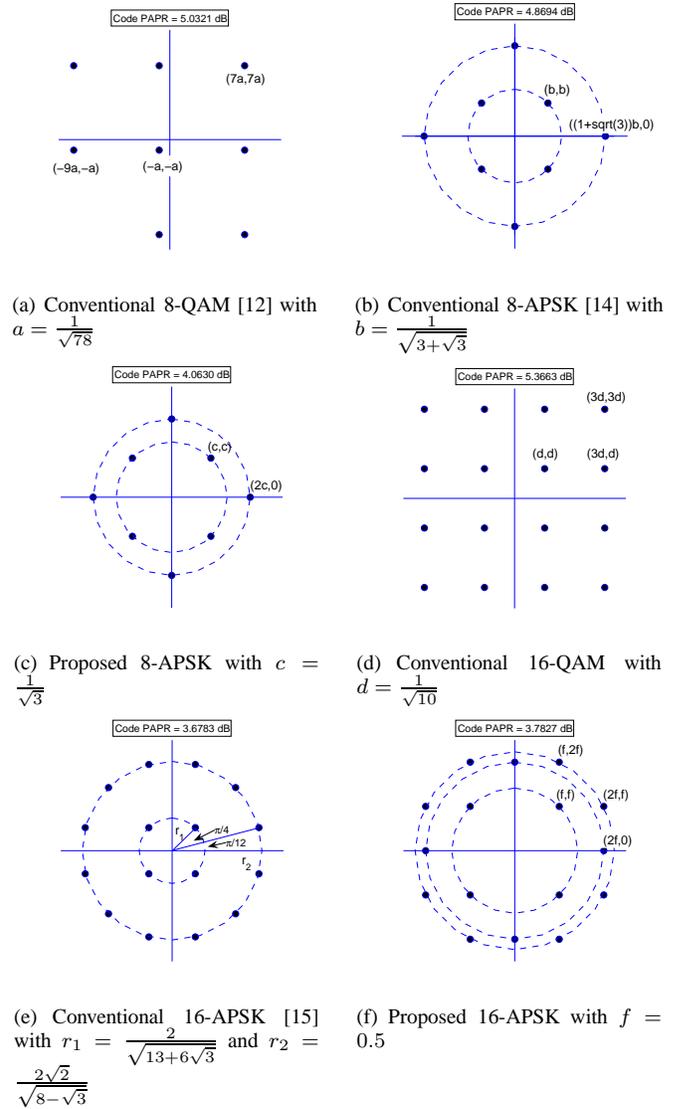


Fig. 3. QAM, conventional APSK and the proposed APSK with unit-average-power symbols ($\sqrt{\cdot}$ means $\sqrt{\cdot}$).

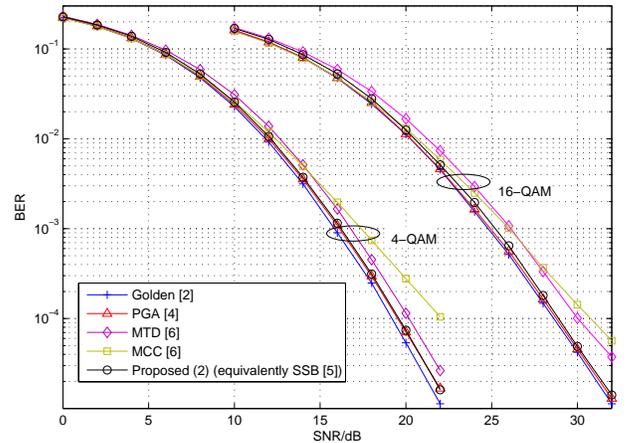


Fig. 4. ML decoding performances of different full-rate codes in 2×2 MIMO systems with 4-QAM and 16-QAM constellations.

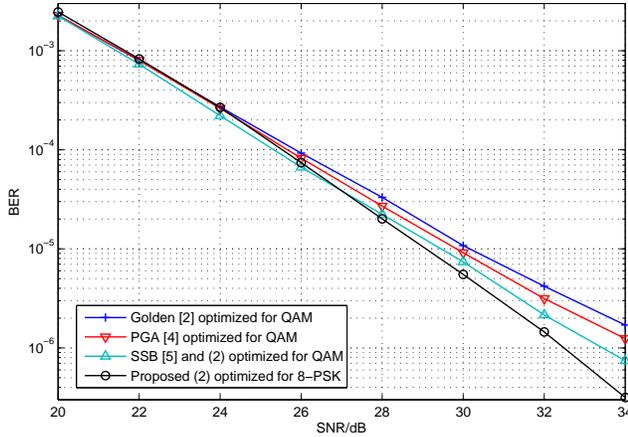


Fig. 5. ML decoding performances of different full-rate codes in 2×2 MIMO systems with 8-PSK constellation.

B. Non-Integer-Coordinate Signal Constellations

The BER performance of the code \mathbf{X} in (2) is next compared with other full-rate codes [2] [4] [5] with 8-PSK in Fig. 5. Here the design coefficients in (14) are adopted for the code \mathbf{X} in (2), while the optimum design coefficients for the other codes are taken from their respective publications. From the simulation results, we can see that the code \mathbf{X} in (2) achieves a larger BER slope when the SNR is high. This is because the other codes, including Golden code, were optimized for QAM, not PSK.

The coding gains of the full-rate STBC's with QAM and PSK constellations are tabulated in Table I with the average power of information symbols normalized to 1. In all cases, they concur with the BER observations made in Fig. 4 and 5.

TABLE I
CODING GAIN COMPARISONS (CONSIDERING
UNIT-AVERAGE-POWER INFORMATION SYMBOLS).

2×2 STBC	4-QAM	16-QAM	8-PSK ^a
Golden [2]	3.2	0.128	
PGA [4]	2.286	0.0914	
SSB [5]	2	0.08	
MTD [6]	0.64	0.0022	
MCC [6]	Non-full diversity	Non-full diversity	
Proposed code \mathbf{X} in (2)	2	0.08	0.0288

^a In [2]–[6], the code design coefficients for 8-PSK are not given.

C. APSK Constellations

Comparisons of the properties and performance of the code \mathbf{X} in (2), when used with the conventional APSK topology versus the proposed APSK topology shown in Fig. 3, are presented in Table II and Fig. 6, respectively. Note that Fig. 3(b) is the best known conventional 8-APSK (in SISO sense), while Fig. 3(e) is the 16-APSK adopted by the DVB-S2 Standard [15]. In the BER simulations, the corresponding optimum code design coefficients r 's from Table II are applied. Note from Table II that the proposed APSK does not need to change its design coefficient r for different constellation

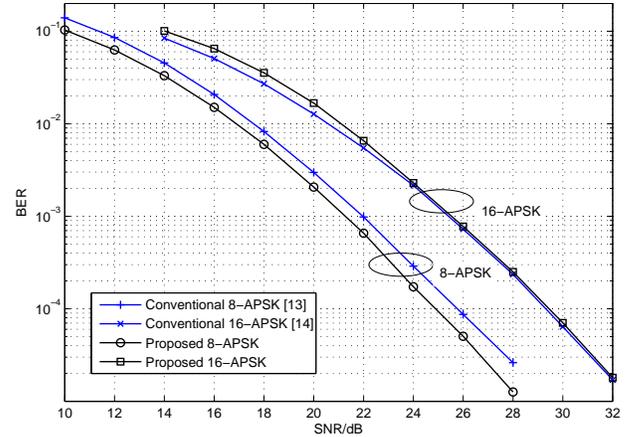


Fig. 6. ML decoding performances of the code \mathbf{X} in (2) in 2×2 MIMO systems with the conventional and proposed 8/16-APSK constellations shown in Fig. 3.

dimensions as it achieves non-vanishing determinant, but this is not true for the conventional APSK.

Interestingly, Table II shows that although the proposed APSK shown in Fig. 3(c) and Fig. 3(f) have smaller minimum Euclidean distance (hence lower PAPR for the proposed 8-APSK), they achieve higher coding gain than the conventional APSK. This is because the coding gains do not depend linearly nor solely on the minimum Euclidean distance, as shown in (9) and (11).

Fig. 6 shows that the code \mathbf{X} in (2) with the proposed 8-APSK has much better performance than the conventional 8-APSK, while the proposed 16-APSK has similar performance as the conventional 16-APSK at high SNR. Fig. 6 also testifies that the code design coefficients shown in Table II for the conventional 8/16 APSK achieve full diversity.

TABLE II
COMPARISONS OF CONVENTIONAL AND PROPOSED APSK FOR THE
CODE \mathbf{X} IN (2) (CONSIDERING UNIT-AVERAGE-POWER
INFORMATION SYMBOLS).

APSK	Minimum Euclidean dis.	Code design coefficient ^b : r	Coding gain
Conventional 8-APSK [14]	0.9194	$0.9454 + j0.3258$	0.0230
Proposed 8-APSK	0.8165	$0.9114 + j0.4114$	0.2222
Conventional 16-APSK [15]	0.5848	$0.8294 + j0.5587$	0.0004
Proposed 16-APSK	0.5	$0.9114 + j0.4114$	0.03125

^b For conventional APSK, the code design coefficient r is optimized following the optimization methodology shown in Section II-B; For the proposed APSK, the code design coefficient $r = (1 + \sqrt{7})/4 + j(-1 + \sqrt{7})/4 \approx 0.9114 + j0.4114$ from (12) is used.

IV. CONCLUSIONS

In this paper, a one-parameter full-rate STBC code structure with fast ML decoding capability adapted from [5], $\mathbf{X} = \begin{bmatrix} s_1 + r s_3 & j r^* s_2^* - s_4^* \\ s_2 + r s_4 & -j r^* s_1^* + s_3^* \end{bmatrix}$, is analyzed for non-vanishing determinant. When used with integer-coordinate signal constellations

such as rectangular QAM, the code design coefficient r is analytically optimized to achieve maximum non-vanishing determinants. When used with non-integer coordinate constellations such as M -PSK, the STBC is found to have vanishing determinants even when the minimum Euclidean distance is fixed at 1. For such vanishing-determinant cases, an analytical methodology is presented to optimize the code to achieve maximum coding gain for a specific constellation dimension. In addition, we consider for the first time the use of APSK constellations in the fast-decodable full-rate STBC, and we show that the APSK-STBC can achieve non-vanishing determinant not with the conventional APSK topologies, but with a APSK topology with constellation points lying on square grid and ring radius $\sqrt{m^2 + n^2}$ (m, n integers). The corresponding optimum STBC design coefficient r and non-vanishing coding gain are $(1 \pm \sqrt{7})/4 + j(-1 \pm \sqrt{7})/4$ (or $(-1 \pm \sqrt{7})/4 + j(1 \pm \sqrt{7})/4$) and $1/2$, respectively. BER simulation, coding gain and code PAPR (peak to average power ratio) enumeration results show that the proposed APSK topology leads to lower code PAPR than QAM, and better or similar BER at high SNR.

APPENDIX

Let us first introduce Lemma 2 and Lemma 3 which will be used later to prove Lemma 3 and Lemma 1, respectively. In the following, $b|a$ denotes that b divides a , and $b \nmid a$ denotes that b cannot divide a .

Lemma 2. For integers a, b, c, d and k , if $2^{2k}|a^2 + b^2 + c^2 + d^2$, then

$$\begin{aligned} &\text{either} && 2^{k-1}|a, 2^{k-1}|b, 2^{k-1}|c, 2^{k-1}|d \text{ and} \\ & && 2^k|a, 2^k|b, 2^k|c, 2^k|d, \\ &\text{or} && 2^{k-1}|a, 2^{k-1}|b, 2^{k-1}|c, 2^{k-1}|d \text{ and} \\ & && 2^k \nmid a, 2^k \nmid b, 2^k \nmid c, 2^k \nmid d \end{aligned}$$

will hold.

Proof: The proof is provided by induction on k .

Case $k = 1$: Clearly, $1|a, 1|b, 1|c$ and $1|d$.

As the value of $a^2 \bmod 4$ is equal to 0 or 1 for any integer a and $4|a^2 + b^2 + c^2 + d^2$, one of the following two equations must hold

$$\begin{aligned} a^2 \bmod 4 = b^2 \bmod 4 = c^2 \bmod 4 = d^2 \bmod 4 = 0, \\ a^2 \bmod 4 = b^2 \bmod 4 = c^2 \bmod 4 = d^2 \bmod 4 = 1. \end{aligned}$$

In other words, $2|a, 2|b, 2|c, 2|d$ must hold at the same time, or $2 \nmid a, 2 \nmid b, 2 \nmid c, 2 \nmid d$ must hold at the same time.

Case $k = 2$: Since $16|a^2 + b^2 + c^2 + d^2$, we have $8|a^2 + b^2 + c^2 + d^2$. As the value of $a^2 \bmod 8$ is equal to 1 for any odd integer a , it follows from $8|a^2 + b^2 + c^2 + d^2$ that a, b, c and d are even integers, i.e., $2^{2-1}|a, 2^{2-1}|b, 2^{2-1}|c$ and $2^{2-1}|d$.

Then, $4|(\frac{a}{2})^2 + (\frac{b}{2})^2 + (\frac{c}{2})^2 + (\frac{d}{2})^2$ where $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ and $\frac{d}{2}$ are integers. Applying the conclusions in Case $k = 1$, we have $2^2|a, 2^2|b, 2^2|c, 2^2|d$ at the same time, or $2^2 \nmid a, 2^2 \nmid b, 2^2 \nmid c, 2^2 \nmid d$ at the same time.

Case $k \geq 2$: Now Let $k-1$ be the induction hypothesis, we prove the induction step.

Since $2^{2k}|a^2 + b^2 + c^2 + d^2, 2^{k-2}|a, 2^{k-2}|b, 2^{k-2}|c$ and $2^{k-2}|d$, we have $16|(\frac{a}{2^{k-2}})^2 + (\frac{b}{2^{k-2}})^2 + (\frac{c}{2^{k-2}})^2 + (\frac{d}{2^{k-2}})^2$ where $\frac{a}{2^{k-2}}, \frac{b}{2^{k-2}}, \frac{c}{2^{k-2}}$ and $\frac{d}{2^{k-2}}$ are integers. Following the conclusions in Case $k = 2$, it can be shown that $\frac{a}{2^{k-2}}, \frac{b}{2^{k-2}}, \frac{c}{2^{k-2}}$ and $\frac{d}{2^{k-2}}$ are even integers, i.e., $2^{k-1}|a, 2^{k-1}|b, 2^{k-1}|c$ and $2^{k-1}|d$.

Then, $4|(\frac{a}{2^{k-1}})^2 + (\frac{b}{2^{k-1}})^2 + (\frac{c}{2^{k-1}})^2 + (\frac{d}{2^{k-1}})^2$ where $\frac{a}{2^{k-1}}, \frac{b}{2^{k-1}}, \frac{c}{2^{k-1}}$ and $\frac{d}{2^{k-1}}$ are integers. Applying the conclusions in Case $k = 1$, we have $2|\frac{a}{2^{k-1}}, 2|\frac{b}{2^{k-1}}, 2|\frac{c}{2^{k-1}}, 2|\frac{d}{2^{k-1}}$ at the same time, or $2 \nmid \frac{a}{2^{k-1}}, 2 \nmid \frac{b}{2^{k-1}}, 2 \nmid \frac{c}{2^{k-1}}, 2 \nmid \frac{d}{2^{k-1}}$ at the same time. Hence, $2^k|a, 2^k|b, 2^k|c, 2^k|d$ at the same time, or $2^k \nmid a, 2^k \nmid b, 2^k \nmid c, 2^k \nmid d$ at the same time.

Therefore, Lemma 2 is proved. \blacksquare

Lemma 3. For integers a, b, c, d, e, f, g and h , if $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2 + h^2$ and $2^k|a^2 + b^2 + c^2 + d^2$ where k is an integer, then

$$2^k|ae + bf + cg + dh + af - be + ch - dg.$$

Proof: Since $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2 + h^2$ and $2^k|a^2 + b^2 + c^2 + d^2$, we have $2^{2k}|(a^2 + b^2 + c^2 + d^2)(e^2 + f^2 + g^2 + h^2)$. Let $t_1 = ae + bf + cg + dh, t_2 = af - be + ch - dg, t_3 = ag - bh - ce + df$ and $t_4 = ah + bg - cf - de$, we have

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = (a^2 + b^2 + c^2 + d^2)(e^2 + f^2 + g^2 + h^2).$$

Hence, $2^{2k}|t_1^2 + t_2^2 + t_3^2 + t_4^2$.

From Lemma 2, we have $2^{k-1}|t_1, 2^{k-1}|t_2$. And $2^k|t_1, 2^k|t_2$ at the same time, or $2^k \nmid t_1, 2^k \nmid t_2$ at the same time.

1) When $2^k|t_1, 2^k|t_2$ at the same time, $2^k|t_1 + t_2$;

2) When $2^k \nmid t_1, 2^k \nmid t_2$ at the same time, then $t_1 = 2^{k-1}m_1$ and $t_2 = 2^{k-1}m_2$ where m_1 and m_2 are odd integers. Thus, $(t_1 + t_2) \bmod 2^k = [2^{k-1}(m_1 + m_2)] \bmod 2^k = 2^{k-1}[(m_1 + m_2) \bmod 2] = 0$, i.e., $2^k|t_1 + t_2$.

Combining the two conclusion, $2^k|t_1 + t_2$, i.e., $2^k|ae + bf + cg + dh + af - be + ch - dg$ holds. \blacksquare

In the following, we prove Lemma 1 based on Lemma 3.

In **Case I**, $|\Delta s_1|^2 + |\Delta s_2|^2 = |\Delta s_3|^2 + |\Delta s_4|^2$. For integer-coordinate signal constellations, the difference symbols can be denoted as

$$\begin{aligned} \Delta s_1 &= a + bj, \\ \Delta s_2 &= c + dj, \\ \Delta s_3 &= e + fj, \\ \Delta s_4 &= g + hj \end{aligned}$$

where a, b, c, d, e, f, g and h are integers and will not be zeros at the same time, i.e., $|\Delta s_1|^2 + |\Delta s_2|^2 = a^2 + b^2 + c^2 + d^2 = |\Delta s_3|^2 + |\Delta s_4|^2 = e^2 + f^2 + g^2 + h^2 \neq 0$. Hence, we have

$$\begin{aligned} \tilde{d}_1 &= (|\Delta s_1|^2 + |\Delta s_2|^2)(u - v) \\ &= (a^2 + b^2 + c^2 + d^2)(u - v), \\ \tilde{d}_2 &= (\Delta s_1 \Delta s_3^* + \Delta s_2 \Delta s_4^*)^R - (\Delta s_1 \Delta s_3^* + \Delta s_2 \Delta s_4^*)^I \\ &= ac + bf + cg + dh + af - be + ch - dg. \end{aligned}$$

Let $a^2 + b^2 + c^2 + d^2$ be expressed as $a^2 + b^2 + c^2 + d^2 = 2^k m$ where k is a non-negative integer and m is an odd integer. Following Lemma 3, it can be shown that $2^k|ac + bf + cg +$

$dh + af - be + ch - dg$, i.e., $2^k|\tilde{d}_2$. Hence, $\tilde{d}_2 = 2^k n$ where n is an integer and we have

$$\begin{aligned} & \min_{\Delta_{s_1} \text{ to } \Delta_{s_4}} |\tilde{d}_1 - \tilde{d}_2| \\ &= \min_{m,n} |2^k m(u-v) - 2^k n| \\ &= \min_{m,n} (2^k |m(u-v) - n|) \\ &= \min_{m,n} |m(u-v) - n|. \end{aligned}$$

Since n is an integer decided by \tilde{d}_2 and m is an odd integer, we have $\min_{m,n} |m(u-v) - n| \leq 1/2$. The equality holds if and only if $u-v = \pm 1/2$. Since the coding gain of \mathbf{X} in **Case I** is $\min_{\Delta_{\mathbf{X}}} [\det(\Delta_{\mathbf{X}} \cdot \Delta_{\mathbf{X}}^H)] = \min_{\Delta_{s_1} \text{ to } \Delta_{s_4}} (2|\tilde{d}_1 - \tilde{d}_2|^2)$, it is easy to see that $\min_{\Delta_{\mathbf{X}}} [\det(\Delta_{\mathbf{X}} \cdot \Delta_{\mathbf{X}}^H)] \leq 1/2$ and the equality holds if and only if $u-v = \pm 1/2$.

Hence, Lemma 1 is proved.

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